Basics of Complexity Analysis: The RAM Model and the Growth of Functions

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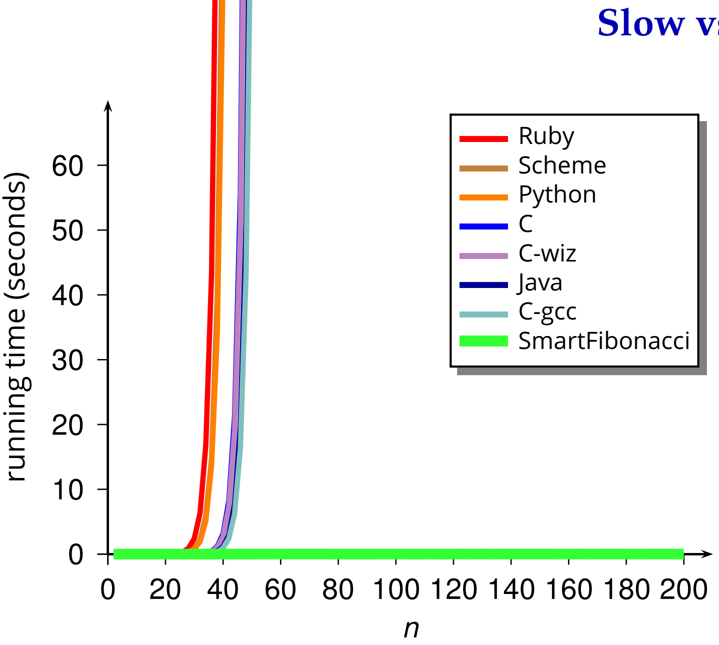
Outline

- Informal analysis of two Fibonacci algorithms
- The *random-access machine* model
- Measure of complexity
- Characterizing functions with their asymptotic behavior
- Big-*O*, omega, and theta notations

Slow vs. Fast Fibonacci



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  - in a way that is ***specific to the algorithms***
  - but ***independent of implementation details***

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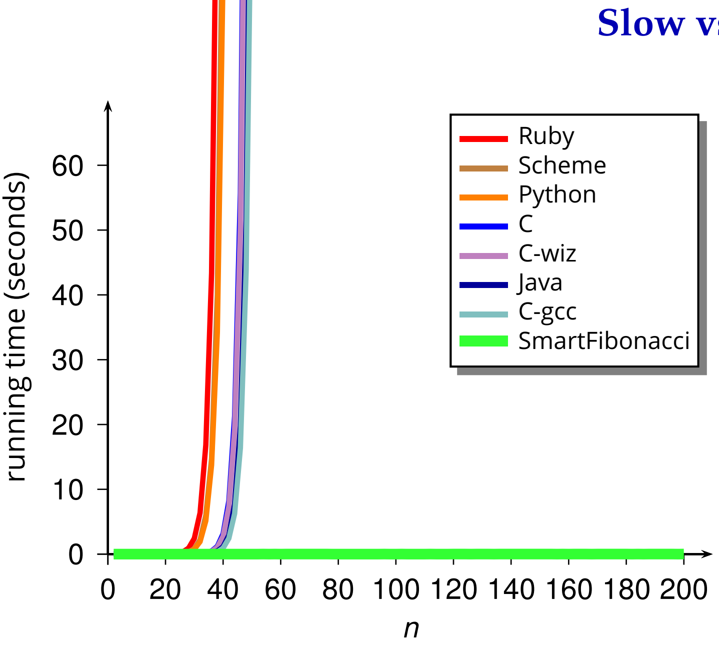

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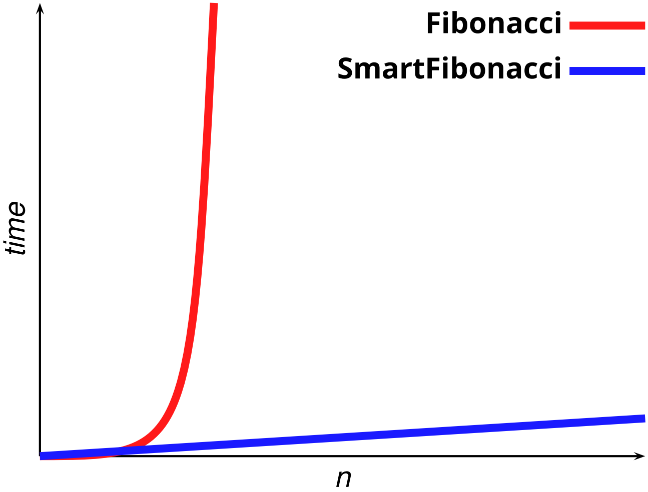

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  - load/store: assignment, use of a variable
  - arithmetic operations: addition, multiplication, division, etc.
  - branch operations: conditional branch, jump
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- A ***basic step*** in the RAM model takes a ***constant time***

```
SmartFibonacci(n)
if n == 0
    return 0
elseif n == 1
    return 1
else pprev = 0
    prev = 1
    for i = 2 to n
        f = prev + pprev
        pprev = prev
        prev = f
return f
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cost	*times* $(n > 1)$
c_1	1
c_2	0
c_3	1
c_4	0
c_5	1
c_6	1
c_7	n
c_8	$n-1$
c_9	$n-1$
c_{10}	$n-1$
c_{11}	1

$$T(n) = c_1 + c_3 + c_5 + c_6 + c_{11} + nc_7 + (n-1)(c_8 + c_9 + c_{10})$$

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$T(n) = nC_1 + C_2 \quad \Rightarrow T(n)$ is a *linear function* of n

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Find(A, x)
for i = 1 to length(A)
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$$T(n) = Cn$$

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FindEquals(A)
for i = 1 to length(A) − 1
    for j = i + 1 to length(A)
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$$T(n) = C\frac{n(n-1)}{2}$$

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  - so, we assume $c_1 = c_2 = c_3 = \cdots = c_i$
  - we also ignore the specific *value* c_i, and in fact ***we ignore every constant cost factor***

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    We write

    $$\boxed{T(n) = \Theta(n^2)}$$

    and say that "$T(n)$ is theta of n-squared"

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- If $f(n)$ is such that $f(n) = kA(g(n))$ for all n sufficiently large and for some constant $k > 0$, then we say that

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- When $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ we also write

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  - $\pi(n) = \Omega(1)$ — trivial ***lower bound***
  - $\pi(n) = \Theta(n/\log n)$ — non-trivial ***tight bound***

Characterizing *Unknown* Functions

- The idea of the O, Ω, and Θ notations is very often to characterize a function that is *not completely known*

  Example:

  Let $\pi(n)$ be the number of *primes* less than or equal to n

  What is the asymptotic behavior of $\pi(n)$?

  - $\pi(n) = O(n)$ — trivial ***upper bound***
  - $\pi(n) = \Omega(1)$ — trivial ***lower bound***
  - $\pi(n) = \Theta(n/\log n)$ — non-trivial ***tight bound***

  In fact, the fundamental *prime number theorem* says that

$$\lim_{n\to\infty} \frac{\pi(n) \ln n}{n} = 1$$

Θ-Notation

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- Given a function $g(n)$, we define the *family of functions* $\Theta(g(n))$

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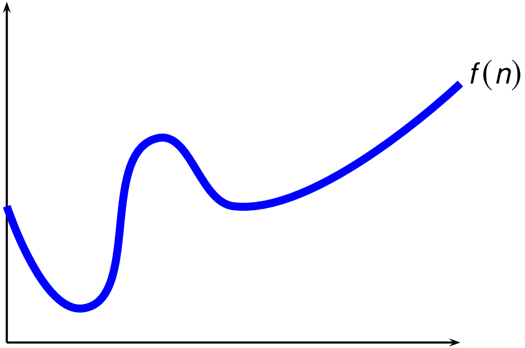

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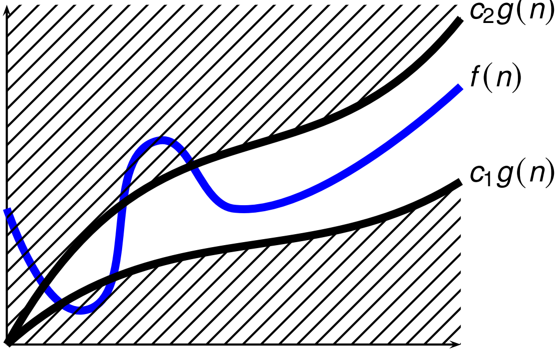

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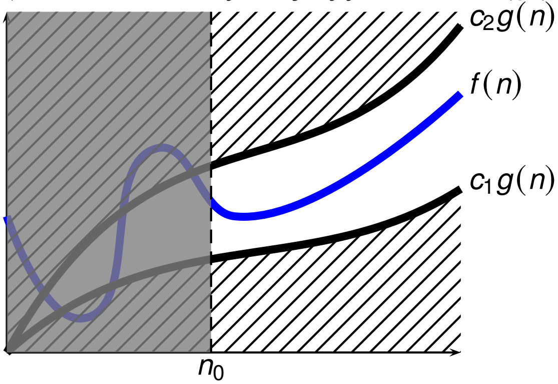

$$\Theta(g(n)) = \{f(n) : \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0 \\ : 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\}$$

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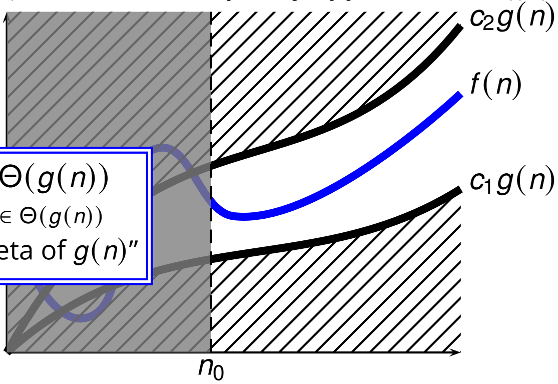

$f(n) = \Theta(g(n))$

i.e., $f(n) \in \Theta(g(n))$

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Examples

- $T(n) = n^2 + 10n + 100$

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- $T(n) =$ complexity of **SmartFibonacci** $\quad \Rightarrow T(n) = \Theta(n)$
- We characterize the behavior of $T(n)$ *in the limit*
- The Θ-notation is an ***asymptotic notation***

O-Notation

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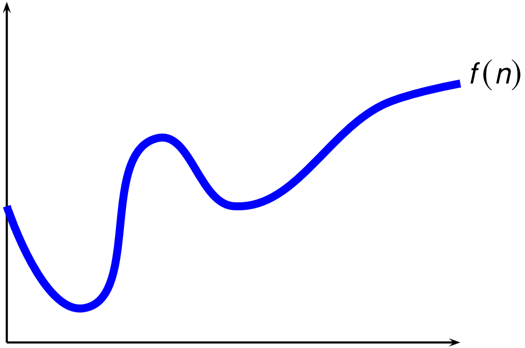

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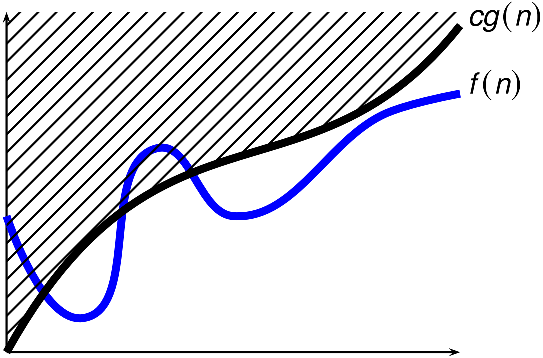

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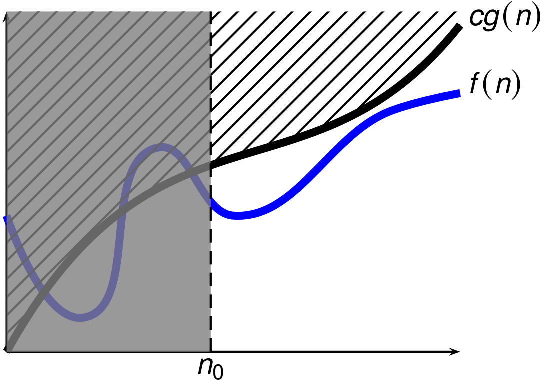

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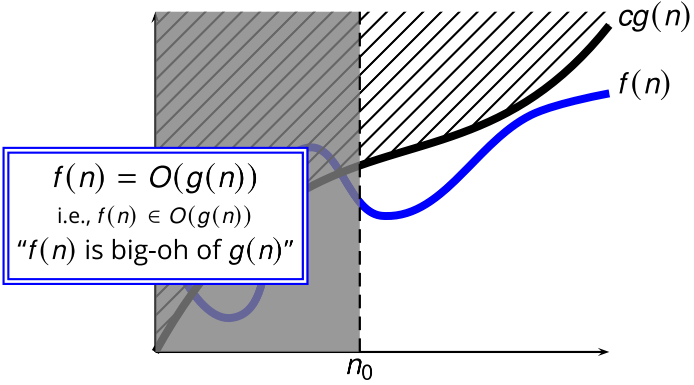

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Example

- So, what is the complexity of **FindEquals**?

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for i = 1 to length(A) − 1
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$$T(n) = \Theta(n^2)$$

- $n = length(A)$ is the ***size of the input***
- we measure the ***worst-case complexity***

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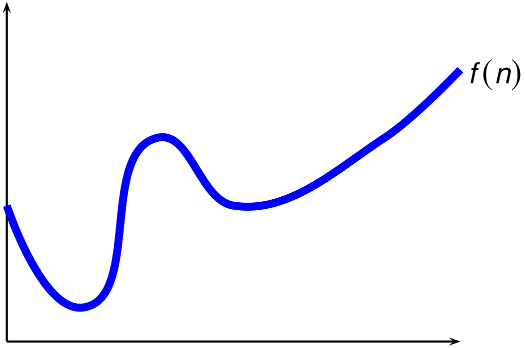

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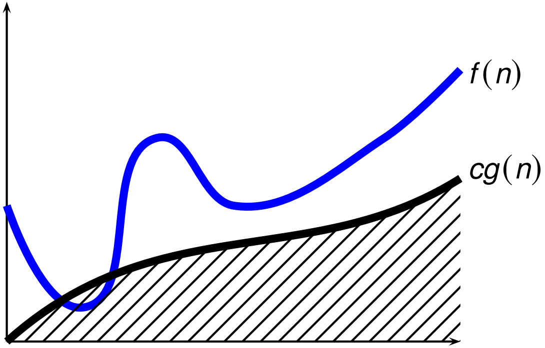

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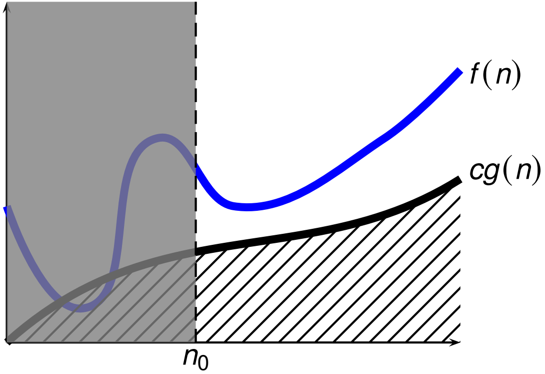

$$\Omega(g(n)) = \{f(n) : \exists c > 0, \exists n_0 > 0 \\ : 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0\}$$

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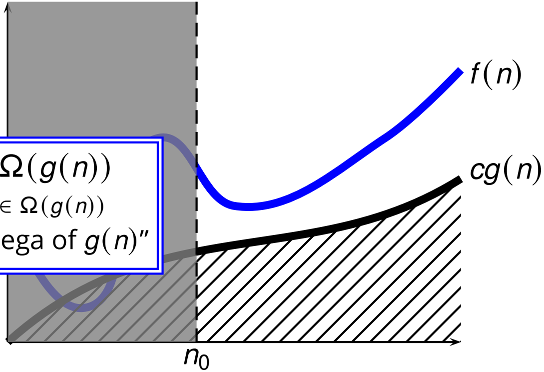

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- *Theorem:* for any two functions $f(n)$ and $g(n)$,
  $f(n) = \Omega(g(n)) \wedge f(n) = O(g(n)) \Leftrightarrow f(n) = \Theta(g(n))$

Θ, O, and Ω as Relations

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- The above theorem can be interpreted as saying

$$f \geq g \wedge f \leq g \Leftrightarrow f = g$$

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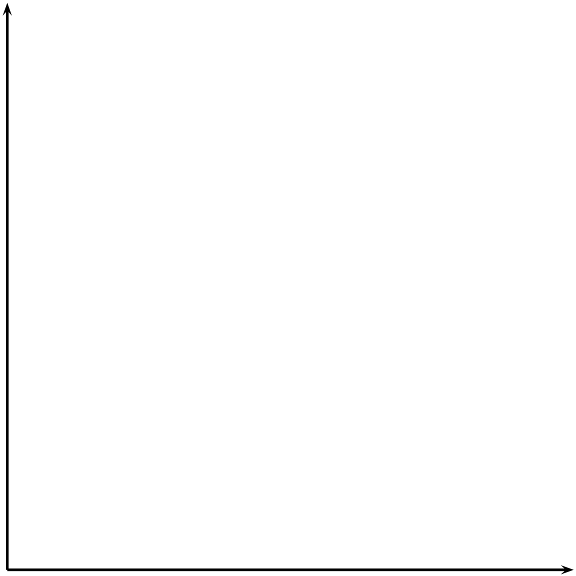
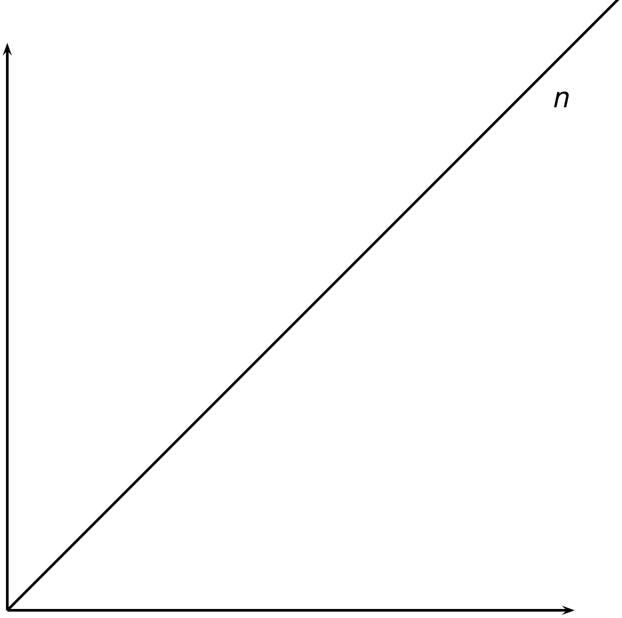
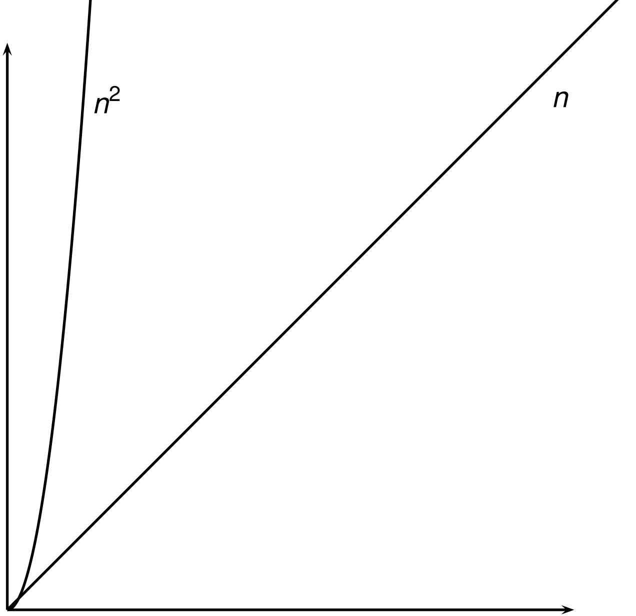
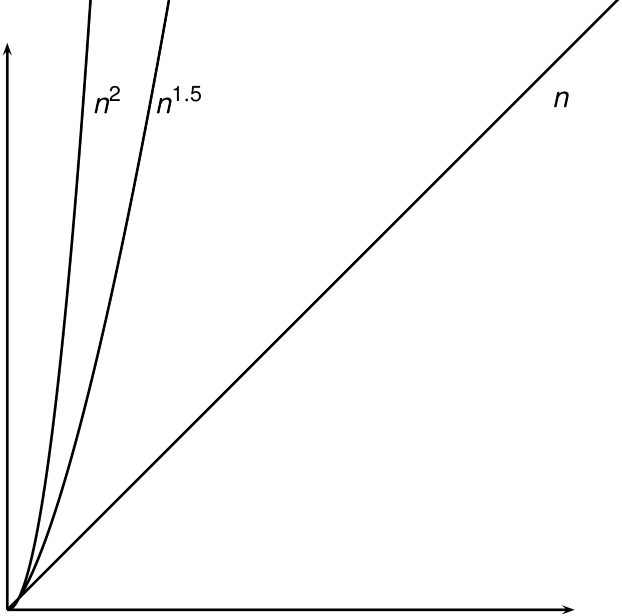


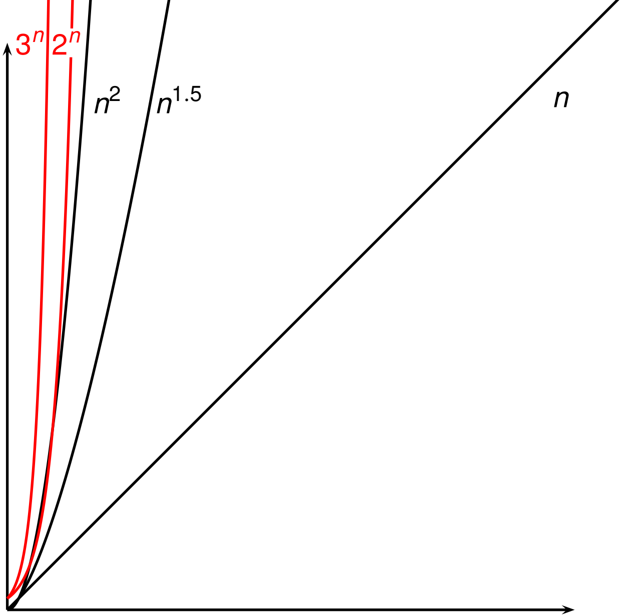

n



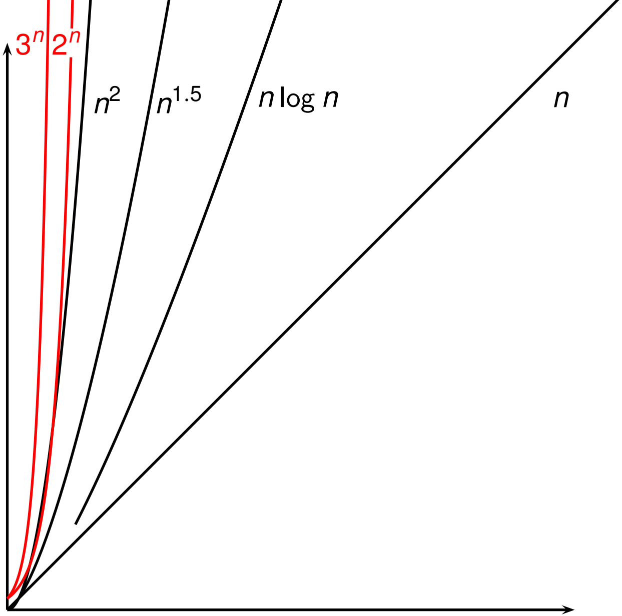
n2
n

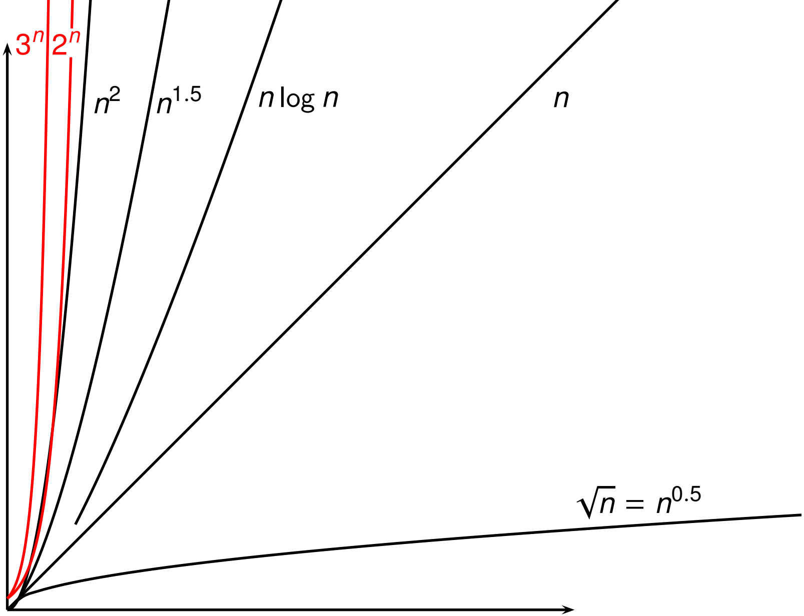


n^2
$n^{1.5}$
n

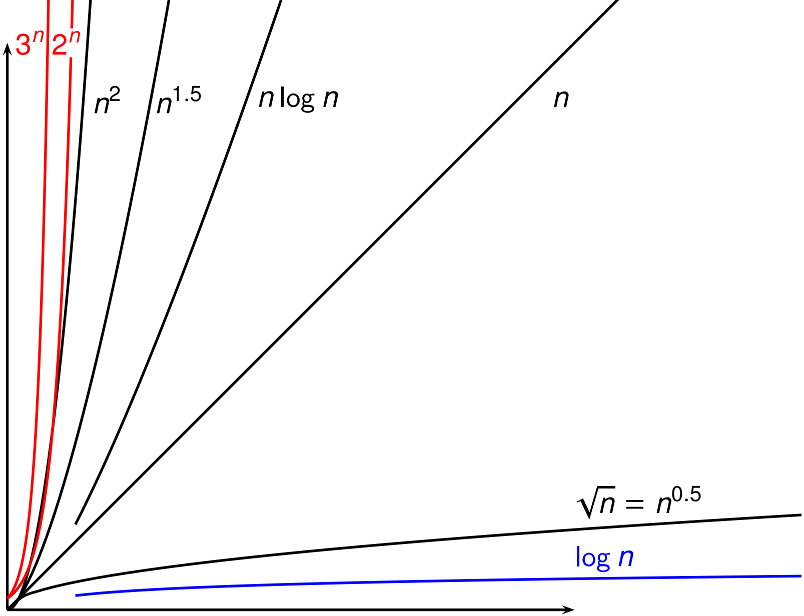


3^n
2^n
n^2
n^1.5
n

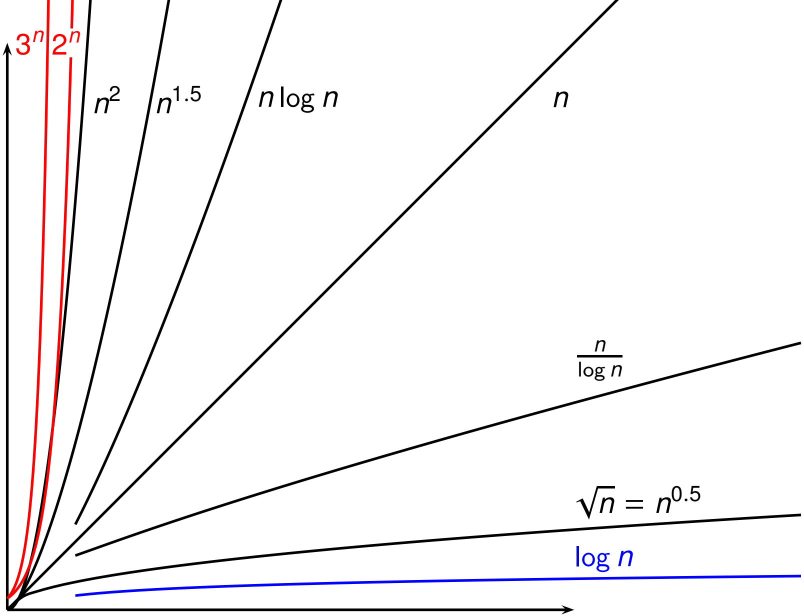

3^n
2^n
n^2
$n^{1.5}$
$n \log n$
n



3^n
2^n
n^2
n^1.5
n log n
n
√n = n^0.5



3^n
2^n
n^2
$n^{1.5}$
$n \log n$
n
$\sqrt{n} = n^{0.5}$
$\log n$



3^n
2^n
n^2
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n
$\frac{n}{\log n}$
$\sqrt{n} = n^{0.5}$
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