Elementary Data Structures and Hash Tables

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Outline

- Common concepts and notation
- Stacks
- Queues
- Linked lists
- Trees
- Direct-access tables
- Hash tables

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  - to facilitate access, or for other purposes

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- A data structure stores ***data*** and possibly ***meta-data***
  - e.g., a *heap* needs an array A to store the keys, plus a variable $A.\mathit{heap\text{-}size}$ to remember how many elements are in the heap

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  - using a linked list
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Push(S,x)

```
S.top = S.top + 1
S[S.top] = x
```

Pop(S)

```
if Stack-Empty(S)
        error "underflow"
else S.top = S.top - 1
        return S[S.top + 1]
```

Queue

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- *Implementation*
  - Q is an array of fixed length $Q.length$
    - i.e., Q holds at most $Q.length$ elements
    - enqueueing more than Q elements causes an “overflow” error
  - $Q.head$ is the position of the “head” of the queue
  - $Q.tail$ is the first empty position at the tail of the queue

Enqueue

Enqueue(Q,x)

```
if Q.queue-full
    error "overflow"
else Q[Q.tail] = x
    if Q.tail < Q.length
        Q.tail = Q.tail + 1
    else Q.tail = 1
    if Q.tail == Q.head
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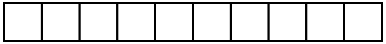

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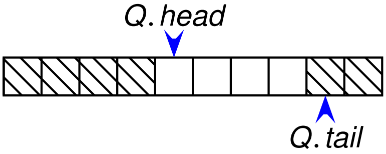

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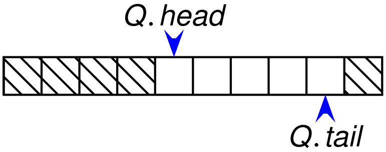

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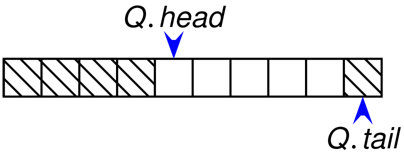

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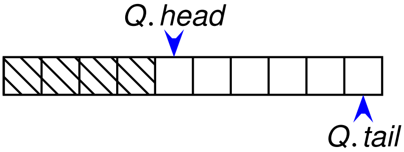

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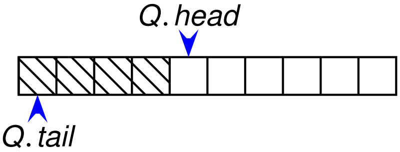

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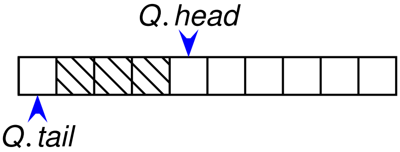

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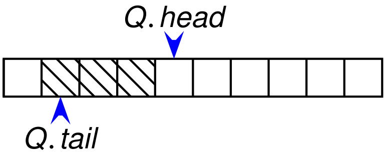

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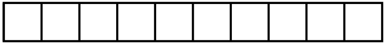

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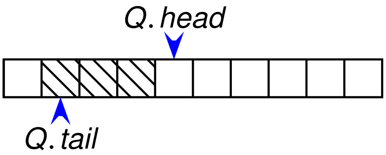

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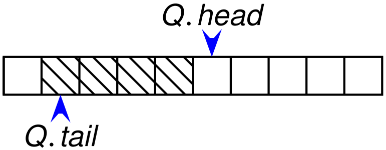

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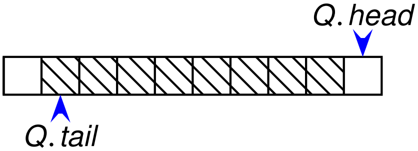

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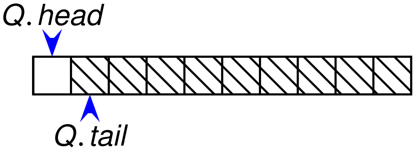

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- *Implementation*
  - a *doubly-linked* list
  - each element x has two “links” $x.prev$ and $x.next$ to the previous and next elements, respectively
  - each element x holds a key $x.key$
  - it is convenient to have a dummy “sentinel” element $L.nil$

Linked List With a "Sentinel"

List-Init(L)

```
L.nil.prev = L.nil
L.nil.next = L.nil
```

List-Insert(L, x)

```
x.next = L.nil.next
L.nil.next.prev = x
L.nil.next = x
x.prev = L.nil
```

List-Search(L, k)

```
x = L.nil.next
while x ≠ L.nil ∧ x.key ≠ k
      x = x.next
return x
```

- *Structure*
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  - unbounded branching

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- *Implementation*
  - for each node $x \neq T.root$, $x.parent$ is x's parent node
  - fixed branching:
    e.g., $x.left\text{-}child$ and $x.right\text{-}child$ in a *binary tree*
  - unbounded branching:
    $x.left\text{-}child$ is x's first (leftmost) child
    $x.right\text{-}sibling$ is x closest sibling to the right

Algorithm	*Complexity*

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List-Insert	$O(1)$
List-Delete	$O(1)$
List-Search	$\Theta(n)$

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Direct-Address-Insert(T, k)
1 $T[k] = \text{true}$

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Direct-Address-Search(T, k)
1 **return** $T[k]$

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- *Can we have the benefits of a direct-address table but with a table of reasonable size?*

- *Idea*
  - use a table T with $|T| \ll |U|$
  - map each key $k \in U$ to a position in T, using a ***hash function***

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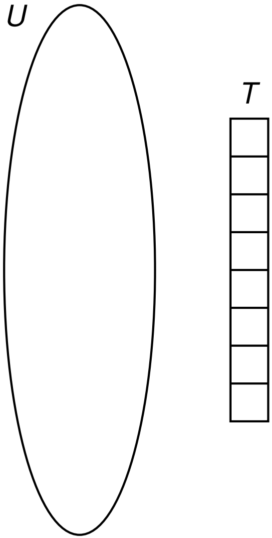
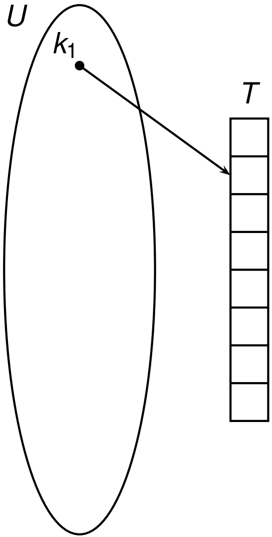
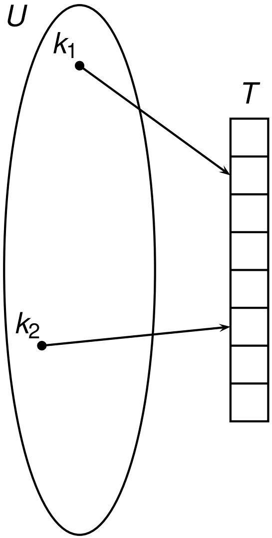
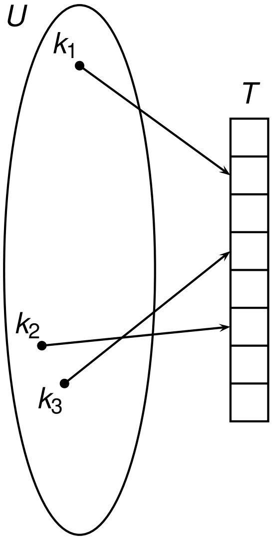
What if two distinct keys $k_1 \neq k_2$ *collide?* (I.e., $h(k_1) = h(k_2)$)


U
T

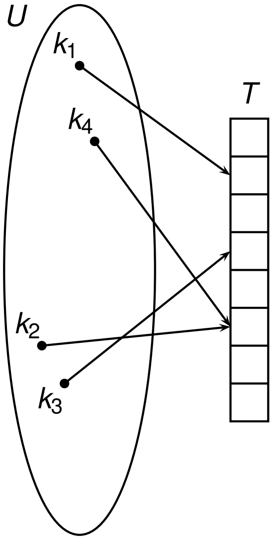


U
k1
T

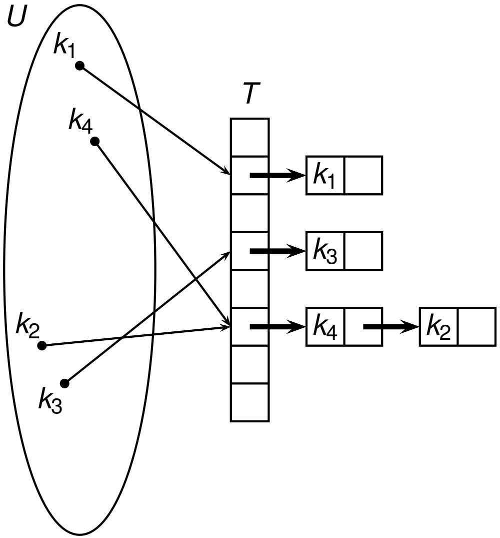

U
k_1
k_2
T

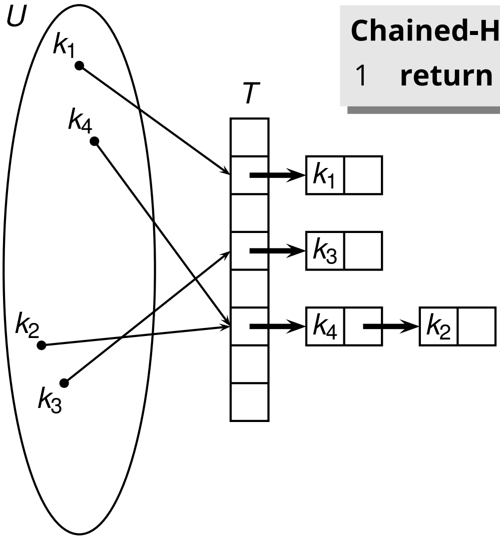


U
k_1
T
k_2
k_3


U
k_1
k_4
k_2
k_3
T

Hash Table





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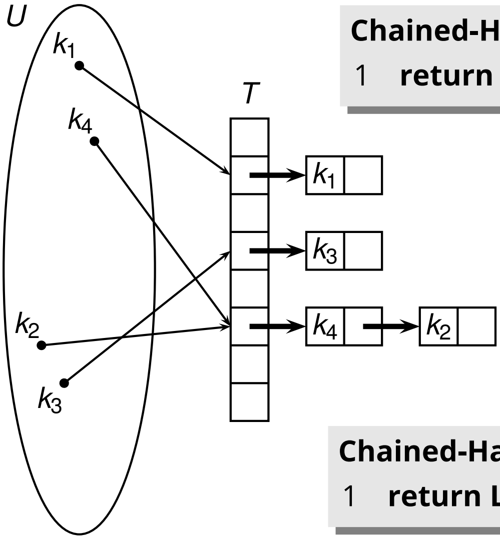

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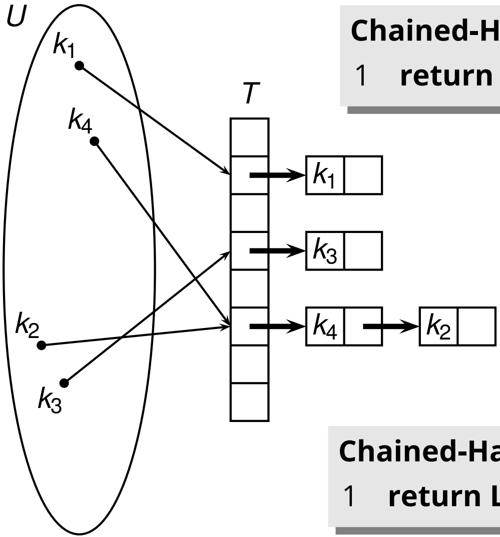

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load factor

$$\alpha = \frac{n}{|T|}$$

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- We further assume that $h(k)$ can be computed in $O(1)$ time

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$$\Pr[h(k) = i] = \frac{1}{|T|} \quad \text{for all } i \in \{1 \dots |T|\}$$

  (The formalism is actually a bit more complicated.)

- So, given n distinct keys, the expected length n_i of the linked list at position i is

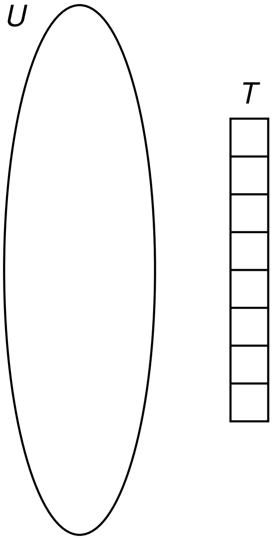
$$\mathrm{E}[n_i] = \frac{n}{|T|} = \alpha$$

- We further assume that $h(k)$ can be computed in $O(1)$ time

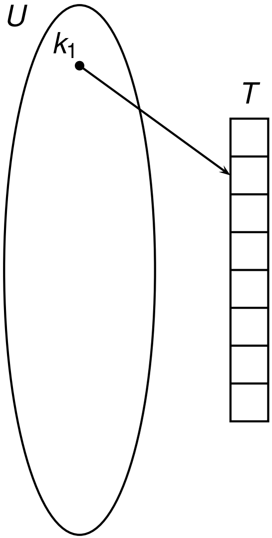
- Therefore, the complexity of **Chained-Hash-Search** is

$$\boxed{\Theta(1 + \alpha)}$$

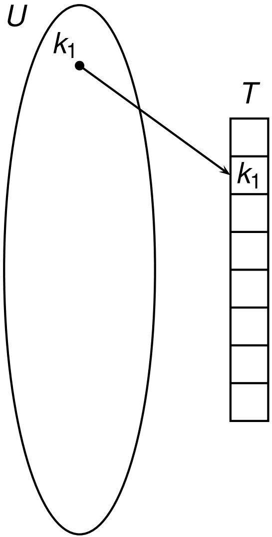
Open-Address Hash Table



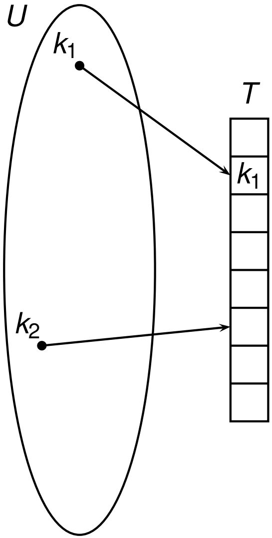
Open-Address Hash Table



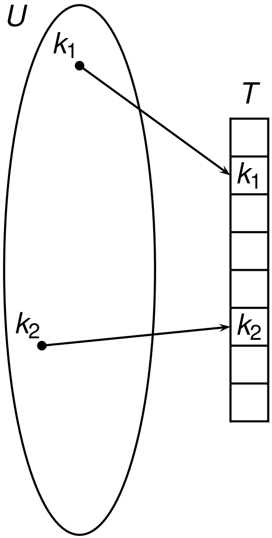
Open-Address Hash Table



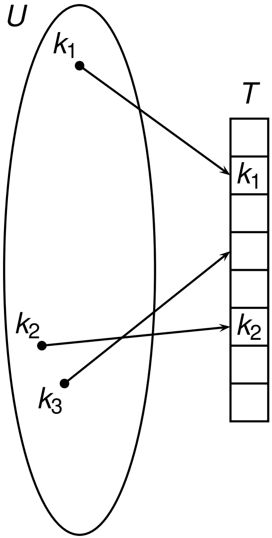
Open-Address Hash Table



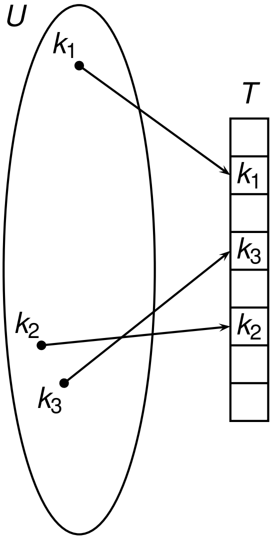
Open-Address Hash Table



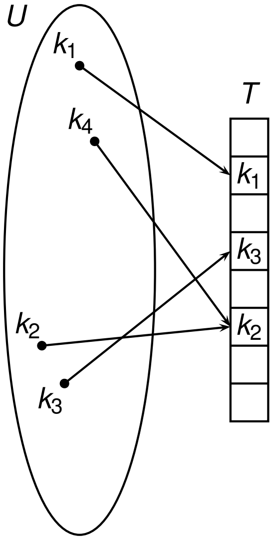
Open-Address Hash Table



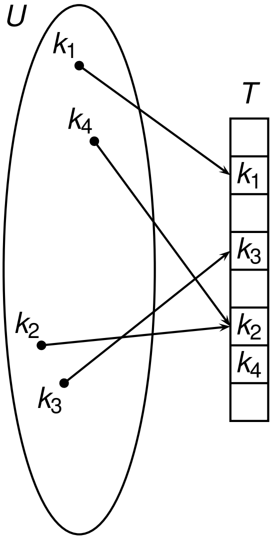
Open-Address Hash Table



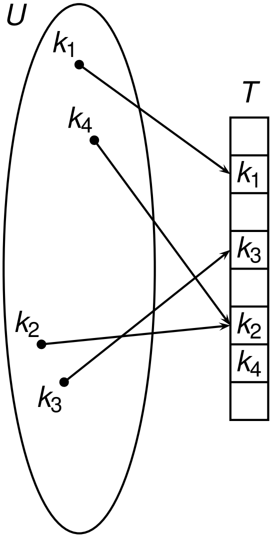
Open-Address Hash Table



Open-Address Hash Table



Open-Address Hash Table



Hash-Insert(T, k)

```
j = h(k)
for i = 1 to T.length
    if T[j] == nil
        T[j] = k
        return j
    elseif j < T.length
        j = j + 1
    else j = 1
error "overflow"
```

- *Idea:* instead of using linked lists, we can store all the elements in the table
  - this implies $\alpha \leq 1$

Open-Addressing (2)

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Open-Addressing (2)

- *Idea:* instead of using linked lists, we can store all the elements in the table
  - this implies $\alpha \leq 1$
- When a collision occurs, we simply find another free cell in T
- A sequential "probe" may not be optimal
  - can you figure out why?

Open-Addressing (3)

```
Hash-Insert(T, k)
for i = 1 to T.length
j = h(k, i)
    if T[j] == nil
        T[j] = k
        return j
error "overflow"
```

```
Hash-Insert(T, k)
for i = 1 to T.length
j = h(k, i)
        if T[j] == nil
                T[j] = k
                return j
error "overflow"
```

- Notice that $h(k, \cdot)$ must be a ***permutation***
  - i.e., $h(k,1), h(k,2), \ldots, h(k,|T|)$ must cover the entire table T