# Dynamic Programming

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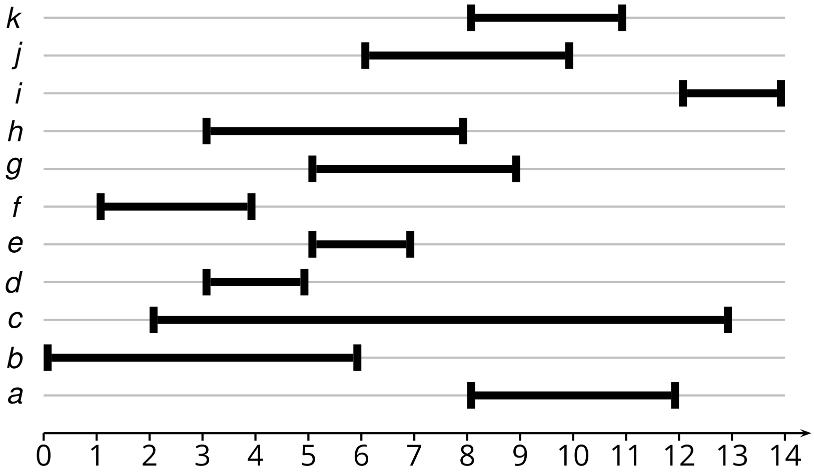
# Outline

- Examples
- Dynamic programming strategy
- More examples

# Activity-Selection Problem



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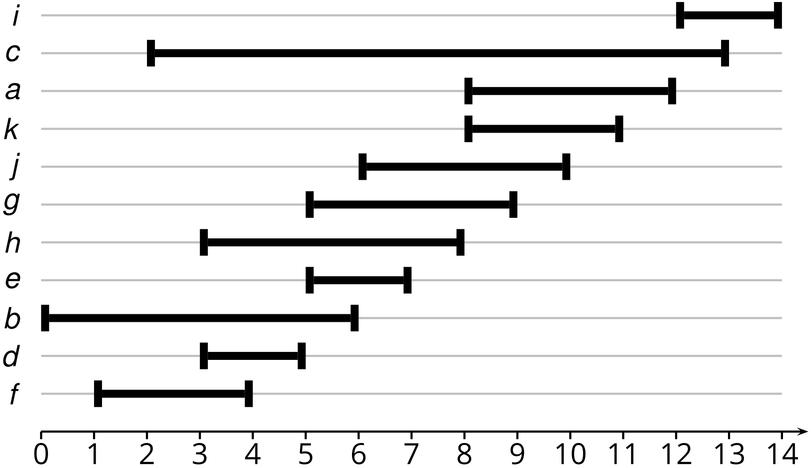


Greedy choice: ***earliest finish time***

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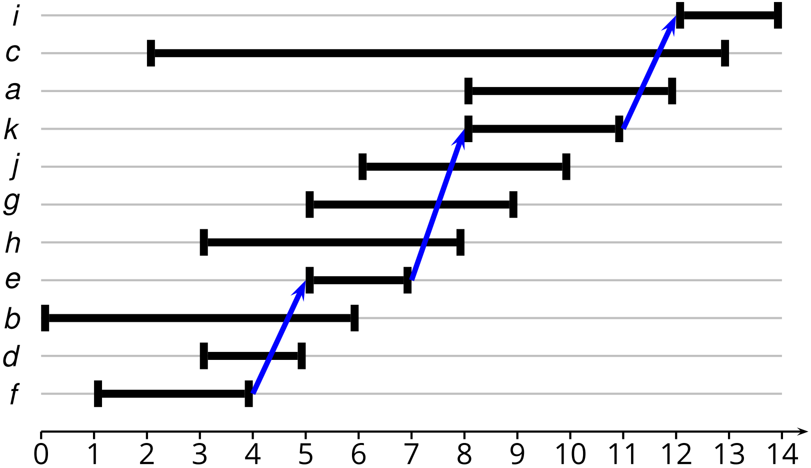


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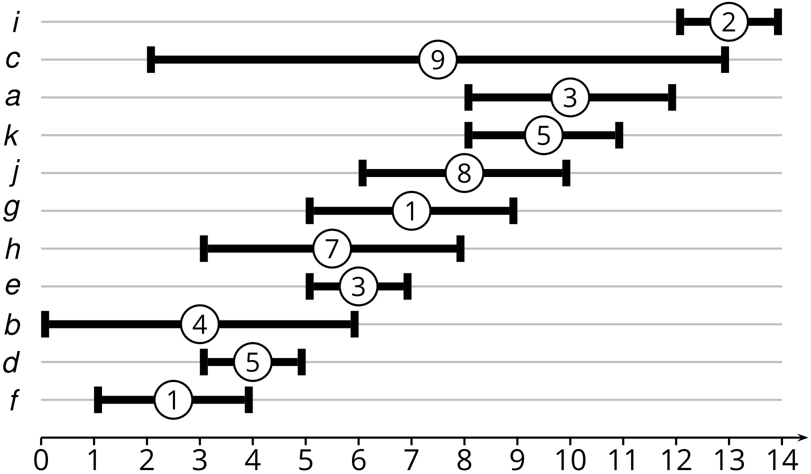

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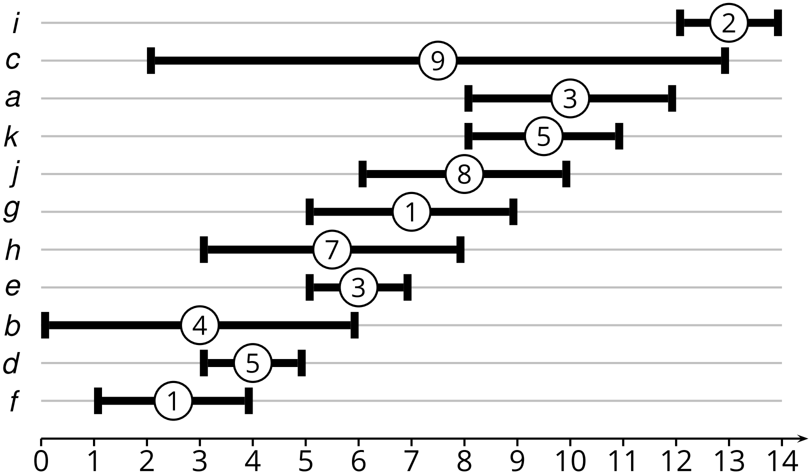
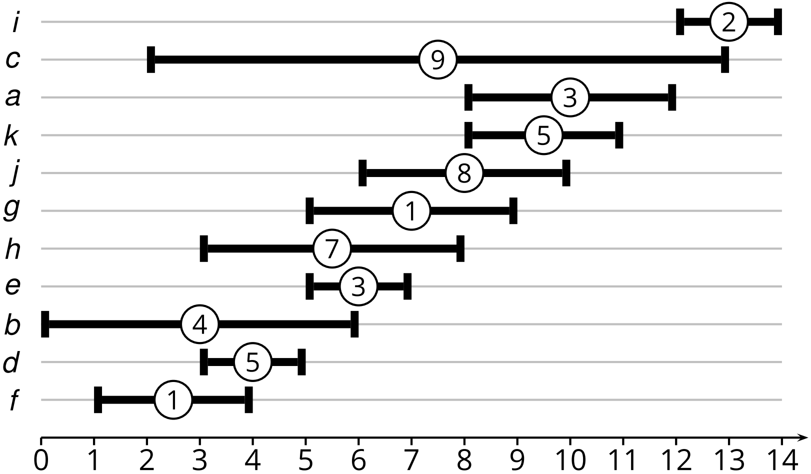



Is the earliest-finish greedy choice still optimal?

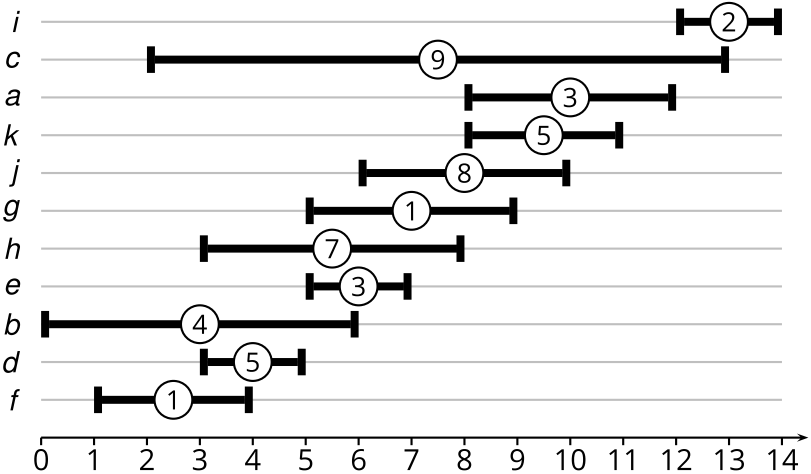
# *Weighted* Activity-Selection Problem

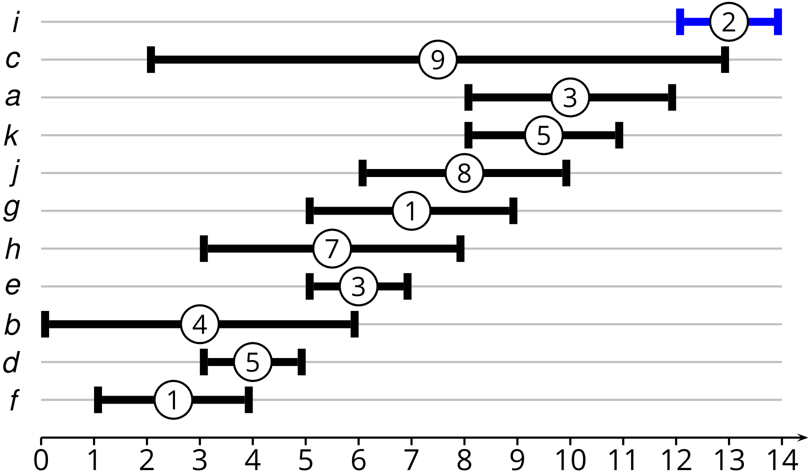



Is the earliest-finish greedy choice still optimal?

Is *any* greedy choice optimal?

**Case 1**






***Case 1:*** activity *i* is in the optimal schedule

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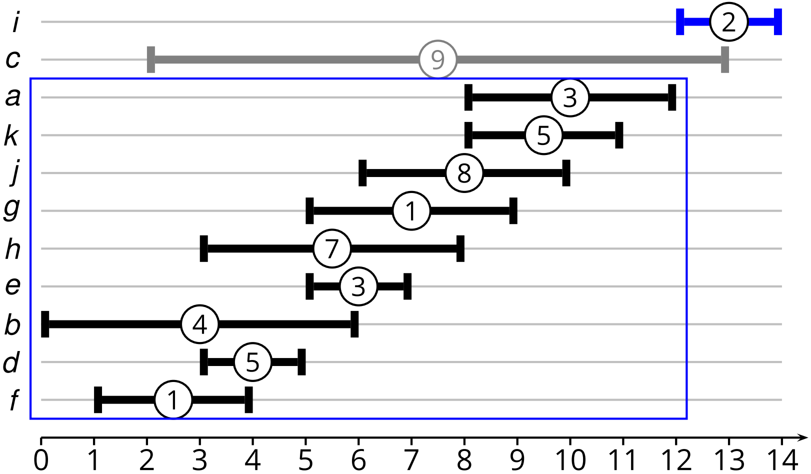


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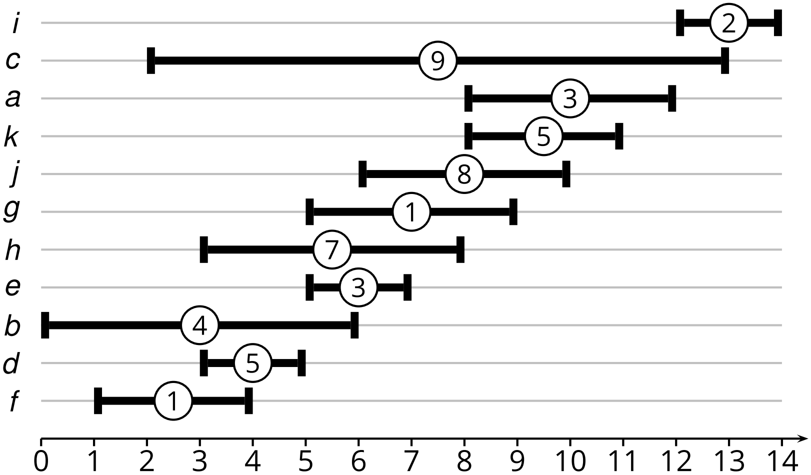
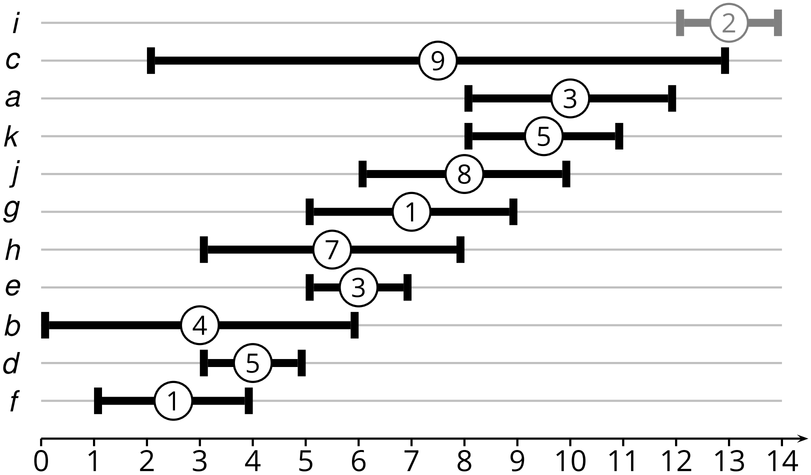
**Case 2**



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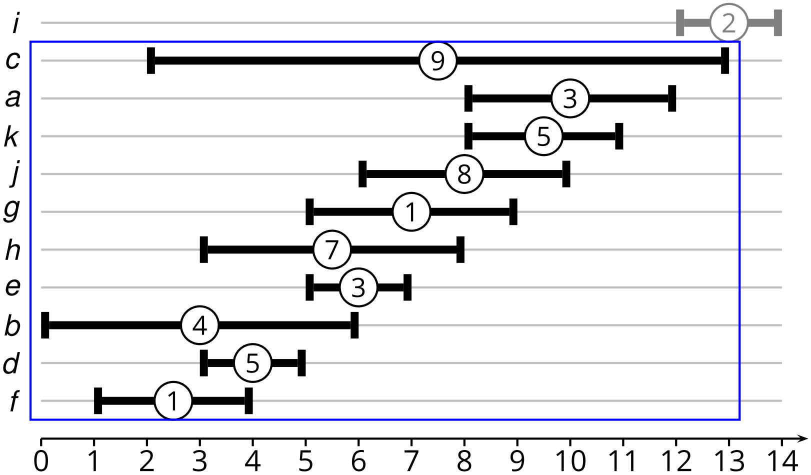


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# Bellman-Ford Algorithm

- Given a graph $G = (V, E)$ and a weight function $w$, we compute the shortest distance $D_u(v)$, from $u \in V$ to $v \in V$, using the *Bellman-Ford equation*

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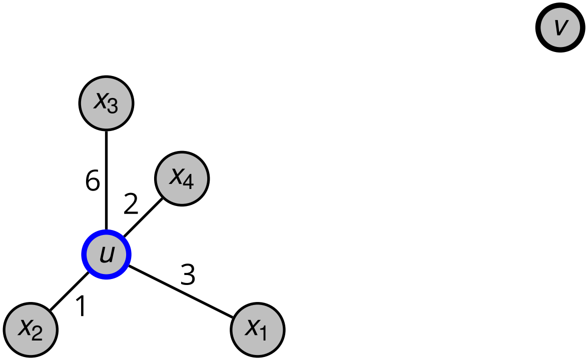

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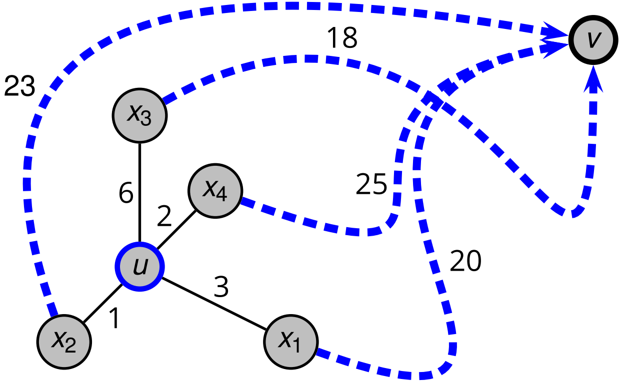

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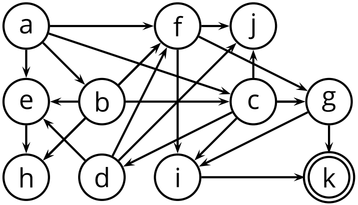

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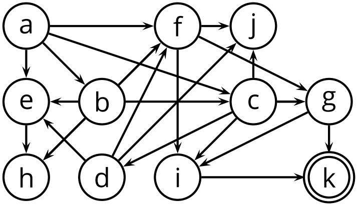
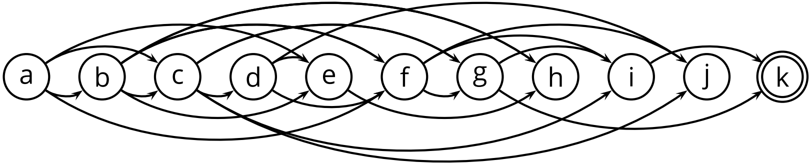



- Considering $V$ in ***topological order***…



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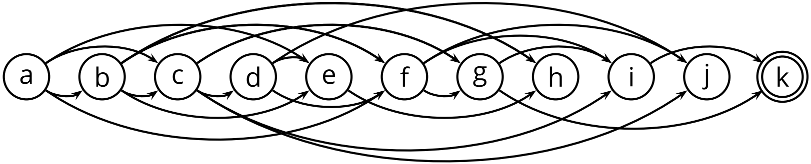

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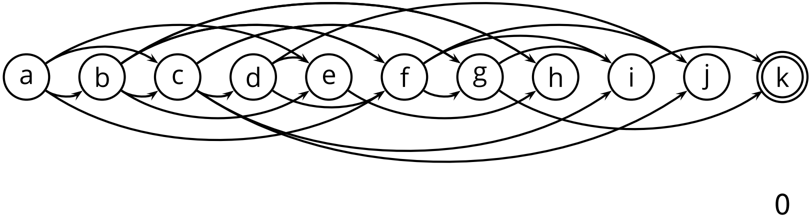

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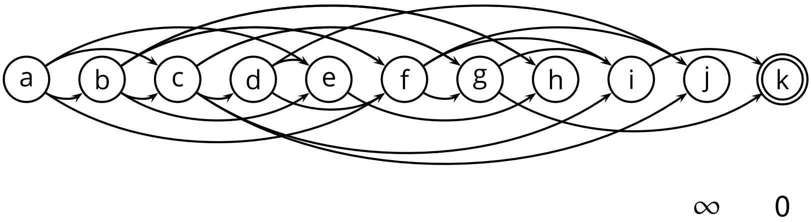

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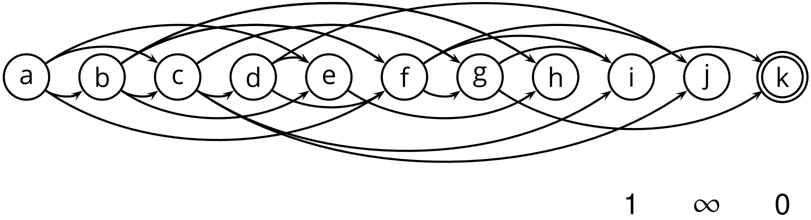

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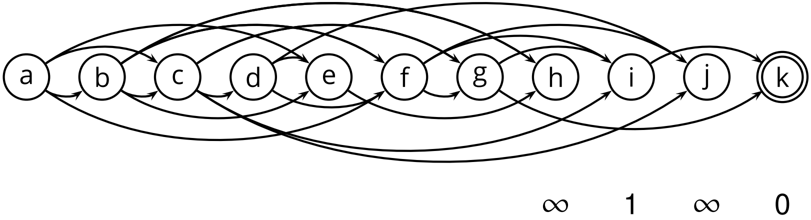

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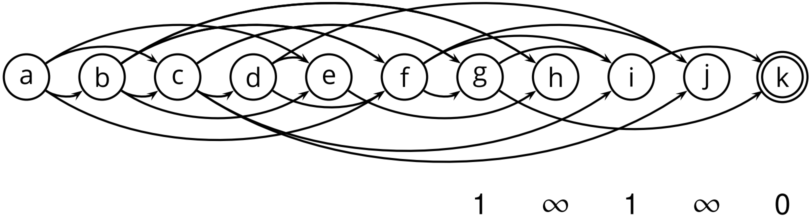

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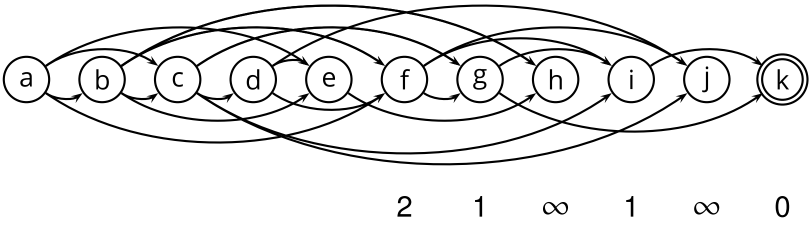

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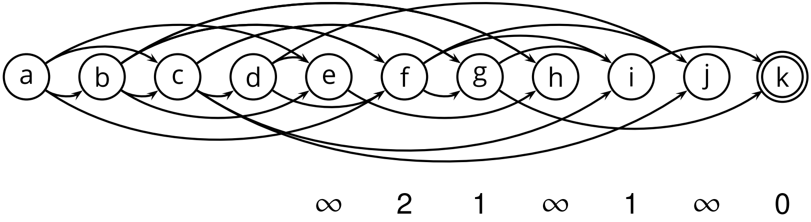

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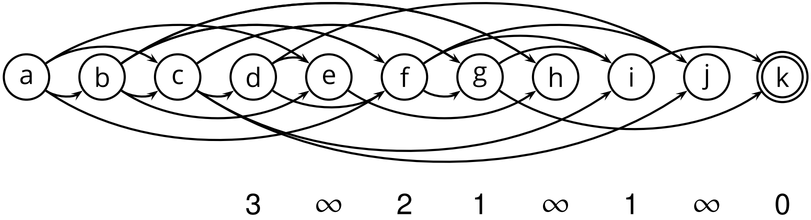

# Shortest Paths on DAGs (2)

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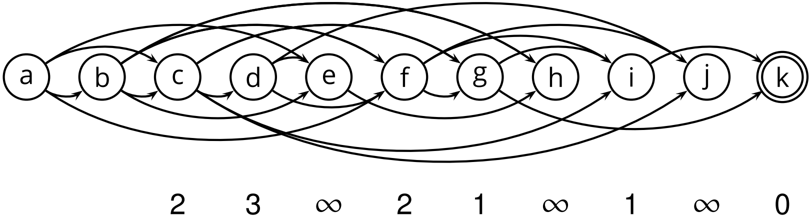

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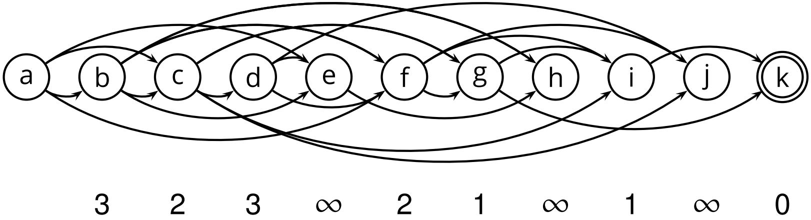

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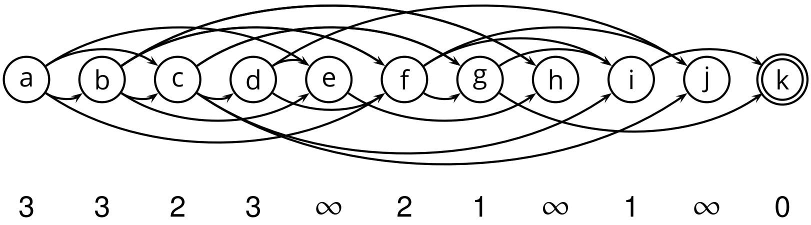

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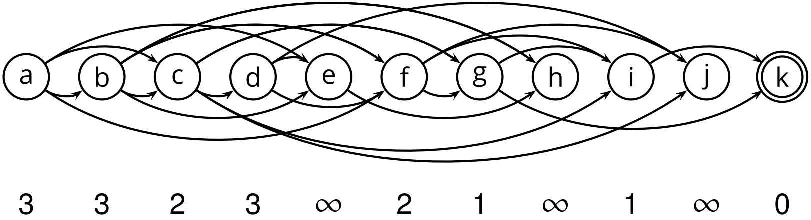


- Since $G$ is a DAG, computing $D_y$ with $y \in Adj(x)$ can be considered a *subproblem* of computing $D_x$
  - we build the solution bottom-up, storing the subproblem solutions

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- Given a sequence of numbers $a_1, a_2, \ldots, a_n$, an *increasing subsequence* is any subset $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, and such that

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A maximal-length subsequence is

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- Combining the subproblems

$$L(j) = 1 + \max\{L(i) \mid i < j \wedge a_i < a_j\}$$

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  - solve the subproblems
  - derive the solution from (one of) the solutions to the subproblems

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  - **exercise:** find a counter-example

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- Divide-and-conquer splits the problem into ***independent subproblems***
  - in dynamic programming, subproblems typically overlap
  - pretty much the same argument as above

# Dynamic Programming vs. Greedy

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  - greedy choice typically *before* proceeding to the subproblem
  - no need to store the result of each subproblem
- Dynamic programming: ***more general***
  - does not need the greedy-choice property
  - typically looks at several subproblems
    - "dynamically" choose one of them to obtain a global solution
  - typically works bottom-up
  - typically reuses solutions of the subproblems

# Typical Subproblem Structures

- Prefix/suffix subproblems
  - *Input:* $x_1, x_2, \ldots, x_n$
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|---|---|---|---|---|---|---|---|---|---|---|
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- Given two strings *x* and *y*, find the *smallest set of edit operations* that transform *x* into *y*
  - edit operations: *delete*, *insert*, and *modify* a single character
  - very important applications
    - spell checker
    - DNA sequencing
- **Example:** transform “Programming Algorithms” into “Jazayeri”

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| ⇓ | | – | | | ⇓ | + | + | | – | – |
| J | | | | | y | e | r | | | |
| C | a | r | z | a | n | | | i | g | a |
| | | | | | | | | | | |
| J | a | | z | a | y | e | r | i | | |

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- Align the two strings $x$ and $y$, possibly inserting “gaps” between letters
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# Edit Distance (3)

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- This suggests a way to combine the subproblems; let $\mathit{diff}(i,j) = 1$ iff $x[i] \neq y[j]$ or 0 otherwise

$$
\begin{aligned}
E(i,j) = \min\{&1 + E(i-1,j), \\
&1 + E(i,j-1), \\
&\mathit{diff}(i,j) + E(i-1,j-1)\}
\end{aligned}
$$

- Problem definition
  - *Input:* a set of $n$ objects with their weights $w_1, w_2, \ldots w_n$ and their values $v_1, v_2, \ldots v_n$, and a maximum weight $W$
  - *Output:* a subset $K$ of the objects such that $\sum_{i \in K} w_i \leq W$ and such that $\sum_{i \in K} v_i$ is maximal

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## Dynamic-programming solution

- let $K(w, j)$ be the maximum value achievable at maximum capacity $w$ using the first $j$ items (i.e., items $1 \ldots j$)
- considering the $j$th element, we can either "use it or loose it," so

$$K(w, j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}$$

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- ***Recursion solves the same problem over and over again***

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- **Idea:** “cache” the results

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Fibonacci(n)
if n == 0
        return 0
elseif n == 1
        return 1
elseif (n, x) ∈ H // a hash table H “caches” results
        return x
else x = Fibonacci(n − 1) + Fibonacci(n − 2)
        Insert(H, n, x)
        return x
```

- Idea also known as *memoization*

# Complexity

- ***Greedy***
    1. start with the greedy choice
    2. add the solution to the remaining subproblem

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  3. in practice, solve the subproblems bottom-up

# Exercise

- **Puzzle 0:** is it possible to insert some '+' signs in the string "213478" so that the resulting expression would equal 214?

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- **Puzzle 0:** is it possible to insert some ‘+’ signs in the string “213478” so that the resulting expression would equal 214?
  - Yes, because $2 + 134 + 78 = 214$
- **Puzzle 1:** is it possible to insert some ‘+’ signs in the strings of digits to obtain the corresponding target number?

| *digits* | *target* |
|---|---:|
| 646805736141599100791159198 | 472004 |
| 6152732017763987430884029264512187586207273294807 | 560351 |
| 48796142803774467559157928 | 326306 |
| 195961521219109124054410617072018922584281838218 | 7779515 |