Divide-and-Conquer Algorithms

Antonio Carzaniga

Faculty of Informatics
Università della Svizzera italiana

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Outline

- Merging (or set union)
- Searching
- Sorting
- Multiplying
- Computing the *median*

Merging (Set Union)

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- **Input:** sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

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- **Example:**

  $A = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10 \rangle$

  $B = \langle 51, 21, 14, 15, 27, 31, 2 \rangle$

  $X =$

- **Input:** sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$

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A Simple Merge Algorithm

- Algorithm strategy

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  - iterate through every position i, first through A, and then B
  - output a_i if a_i is not in $\langle a_1, a_2, \ldots, a_{i-1} \rangle$
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```
MergeSimple(A, B)
for i = 1 to length(A)
    if not Find(A[1 . . i − 1], A[i])
        output A[i]
for i = 1 to length(B)
    if not Find(A, B[i]) and not Find(B[1 . . i − 1], B[i])
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Complexity

MergeSimple(A, B)

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```

let $n = length(A) + length(B)$

$$T(n) = \sum_{i=1}^{length(A)} T_{\mathbf{Find}}(i) + \sum_{i=1}^{length(B)} \left(T_{\mathbf{Find}}(i) + T_{\mathbf{Find}}(length(A))\right)$$

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Find(A, key)
for i = 1 to length(A)
    if A[i] == key
        return true
return false
```

```
Find(A, begin, end, key)
for i = begin to end
    if A[i] == key
        return true
return false
```

Searching

- **Input:** a sequence A and a value *key*

  Output: true if A contains *key*, or false otherwise

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- The complexity of **Find** is

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Searching in a List

- **Input:** a sequence *A* and a value *key*

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FindInList(A, key)
    item = first(A)
    while item ≠ last(A)
        if value(item) == key
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        item = next(item)
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Searching (2)

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BinarySearch(A, key)
first = 1
last = length(A)
while first ≤ last
    middle = ⌈(first + last)/2⌉
    if A[middle] == key
        return true
    elseif first = last
        return false
    elseif A[middle] > key
        last = middle − 1
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Binary Search

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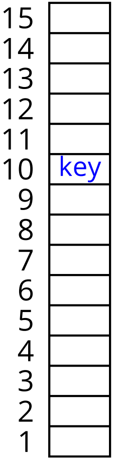

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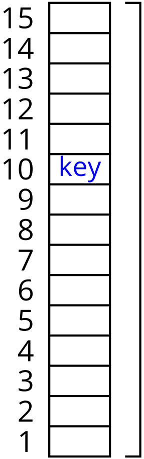

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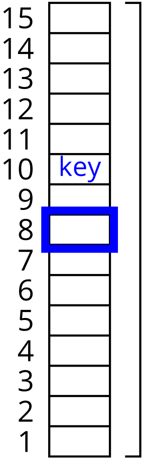

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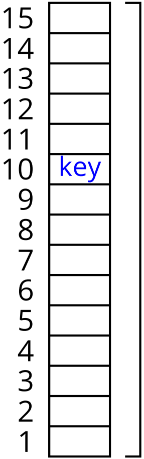

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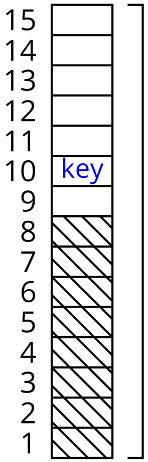

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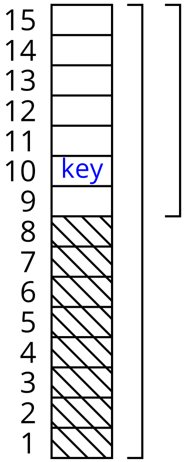

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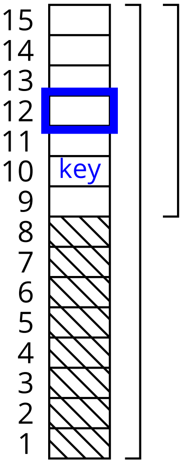

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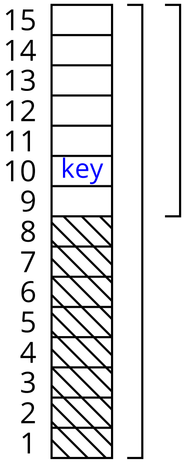

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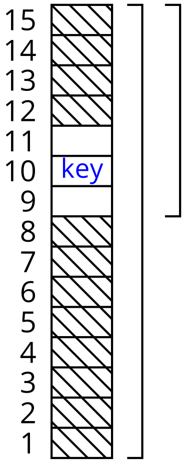

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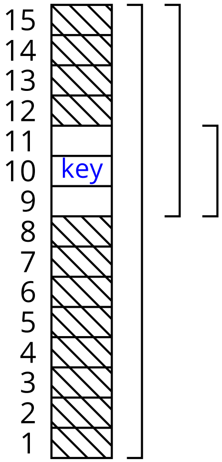

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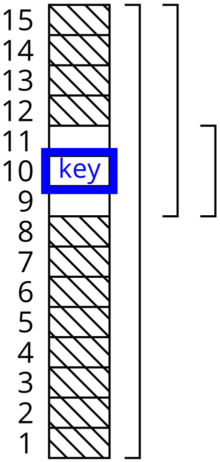

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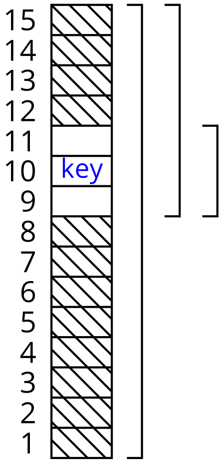

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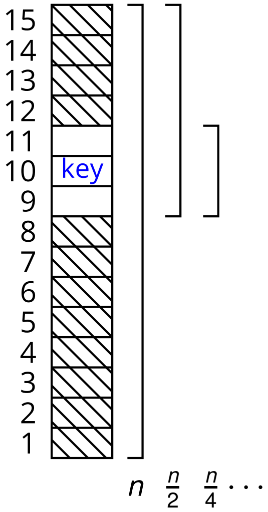

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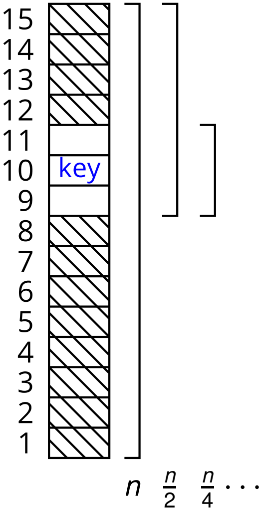

$$T(n) = O(\log n)$$

- A slightly different problem:

  Input: two *sorted* sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$, where $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_m$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

  - every element of A appears once in X
  - every element of B appears once in X
  - every element of X appears in A or in B or in both

A Better Merge Algorithm

```
MergeSimple2(A, B)
for i = 1 to length(A)
        if not BinarySearch(A[1 . . i − 1], A[i])
                output A[i]
for i = 1 to length(B)
        if not BinarySearch(A, B[i])
        and not BinarySearch(B[1 . . i − 1], B[i])
                output B[i]
```

A Better Merge Algorithm

```
MergeSimple2(A, B)
for i = 1 to length(A)
    if not BinarySearch(A[1 .. i − 1], A[i])
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$$T(n) = \sum_{i=1}^{n} O(\log i) = O(n \log n)$$

Better than $O(n^2)$, but can we do even better than $O(n \log n)$?

An Even Better Merge Algorithm

- *Intuition:* A and B are *sorted*
  e.g.
  $A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$
  $B = \langle 1, 5, 6, 7, 34, 35, 40, 41, 43 \rangle$

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  so just like in **BinarySearch** I can avoid looking for an element x if the *first* element I see is $y > x$

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  so just like in **BinarySearch** I can avoid looking for an element x if the *first* element I see is $y > x$

- High-level algorithm strategy
  - step through every position i of A and every position j of B
  - output a_i and advance i if $a_i \leq b_j$ or if j is beyond the end of B
  - output b_j and advance j if $a_i \geq b_j$ or if i is beyond the end of A

Merge Algorithm

A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

Merge Algorithm

$i = 1$

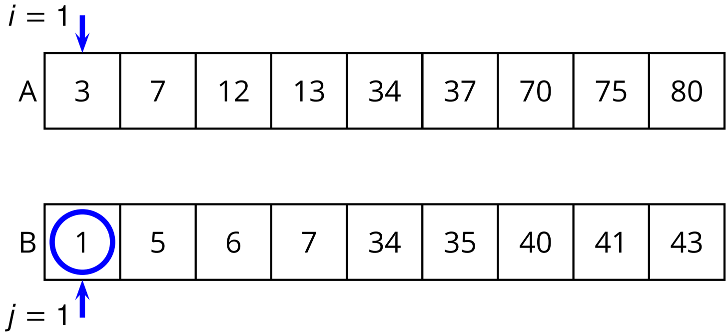
A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 1$

Output:

Merge Algorithm



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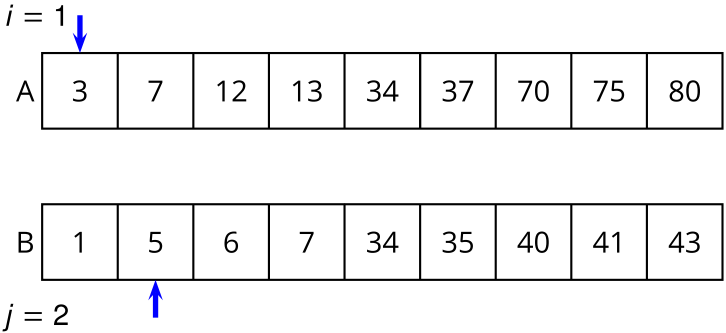


Output: 1

Merge Algorithm

$i = 1$

A	3	7	12	13	34	37	70	75	80
B	**1**	**5**	**6**	**7**	**34**	**35**	**40**	**41**	**43**

$j = 2$

Output: 1

$i = 2$

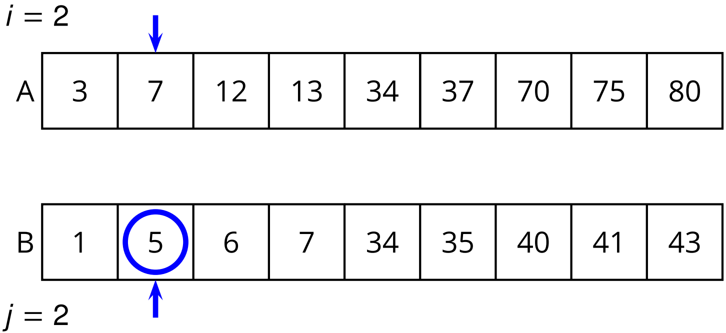
A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 2$

Output: 1 3

Merge Algorithm



Output: 1 3

Merge Algorithm

$i = 2$

A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 3$

Output: 1 3 5

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Merge Algorithm

$i = 5$

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Output: 1 3 5 6 7 12 13...

Merge Algorithm (2)

```
Merge(A, B)
i, j = 1
X = ∅
while i ≤ length(A) or j ≤ length(B)
        if i > length(A)
                X = X ∘ B[j]          // appends B[j] to X
                j = j + 1
        elseif j > length(B)
                X = X ∘ A[i]
                i = i + 1
        elseif A[i] < B[j]
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- This algorithm is incorrect! (Exercise: fix it)

Complexity of Merge

Merge(A, B)

$i, j = 1$
$X = \emptyset$
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    if $i \leq \mathit{length}(A)$ **and** $(j > \mathit{length}(B)$ **or** $A[i] < B[j])$
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- Can we do better? No!
  - we have to output $n = \mathit{length}(A) + \mathit{length}(B)$ elements

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  - this suggests a recursive algorithm

Merge Sort

MergeSort(A)

```
if length(A) == 1
        return A
m = ⌊length(A)/2⌋
A_L = MergeSort(A[1 .. m])
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- Complexity analysis

$$T(n) = T_{\text{divide}} + \sum_{i=1}^{k} T(|A_i|) + T_{\text{combine}}$$

we will analyze this formula another time…

A Divide-and-Conquer Merge

MergeR(A, B)

```
if length(A) == 0
        return B
if length(B) == 0
        return A
if A[1] < B[1]
        return A[1] ∘ MergeR(A[2 . . length(A)], B)
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- Can we do better? No! (We knew that already)

Divide-and-Conquer Multiplication

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  which means $x = 2^{\ell/2}x_L + x_R$ and $y = 2^{\ell/2}y_L + y_R$, so...

$$\begin{aligned} xy &= (2^{\ell/2}x_L + x_R)(2^{\ell/2}y_L + y_R) \\ &= 2^{\ell}x_Ly_L + 2^{\ell/2}(x_Ly_R + x_Ry_L) + x_Ry_R \end{aligned}$$

  we reduced the problem of multiplying two numbers of ℓ bits into the problem of multiplying *four* numbers of $\ell/2$ bits...

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$$T(\ell) = 3T(\ell/2) + O(\ell)$$

which, as we will see, leads to a much better complexity

$$T(\ell) = O(\ell^{\log_2 3}) = O(\ell^{1.59})$$

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- Can we do better? Let's try *divide-and-conquer*...

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  E.g.,

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  - for $k = \lfloor |A|/2 \rfloor$, the median of A
  - what is the *6th smallest* element of $A = \langle 2, 36, 5, 21, 8, 13, 11, 20, 5, 4, 1 \rangle$?

- The *median* of a sequence A is a value $m \in A$ such that half the values in A are less than or equal to m

- Generalizating, the ***k-smallest*** element of a sequence A is a value $v \in A$ such that exactly k elements of A are less than or equal to v

  E.g.,

  - for $k = 1$, the minimum of A
  - for $k = \lfloor |A|/2 \rfloor$, the median of A
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    the 6th smallest element of A—a.k.a. $select(A, 6)$—is 8

k-Smallest Element

- Idea: we split the sequence A in three parts based on a *chosen value* $v \in A$
  - A_L contains the set of elements that are *less than* v
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It is the 2nd smallest value of A_R

k-Smallest Element (2)

We use $select(A, k)$ to denote the k-smallest element of A

$$select(A,k) = \begin{cases} select(A_L, k) & \text{if } k \leq |A_L| \\ v & \text{if } |A_L| < k \leq |A_L| + |A_v| \\ select(A_R, k - |A_L| - |A_v|) & \text{if } k > |A_L| + |A_v| \end{cases}$$

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  - so, ideally we should pick $v = median(A)$, but...
- We pick *a random element of A*

Selection Algorithm

Selection(A, k)

$v = A[\text{random}(1 \ldots |A|)]$
$A_L, A_v, A_R = \emptyset$
for $i = 1$ **to** $|A|$
    if $A[i] < v$
        $A_L = A_L \cup A[i]$
    elseif $A[i] == v$
        $A_v = A_v \cup A[i]$
    else $A_R = A_R \cup A[i]$
if $k \leq |A_L|$
    return Selection(A_L, k)
elseif $k > |A_L| + |A_v|$
    return Selection$(A_R, k - |A_L| - |A_v|)$
else return v