Binary Search Trees

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Outline

- Binary search trees
- Randomized binary search trees

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  - over a ***totally ordered domain***

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  - **Tree-Minimum**(T) finds the smallest element in the tree
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  - *iteration:* **Tree-Successor**(x) and **Tree-Predecessor**(x) find the successor and predecessor, respectively, of an element x

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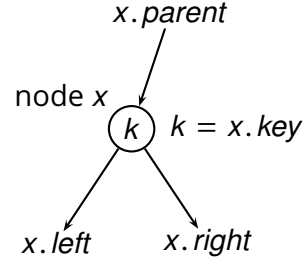
Node x

- $x.parent$ is the parent of node x
- $x.key$ is the key stored in node x
- $x.left$ is the left child of node x
- $x.right$ is the right child of node x



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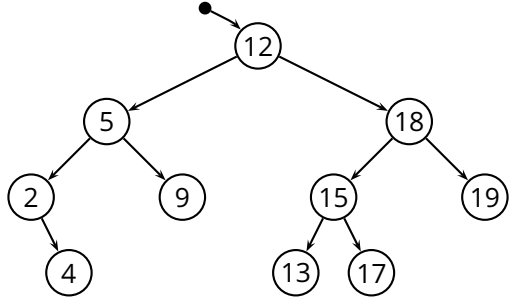

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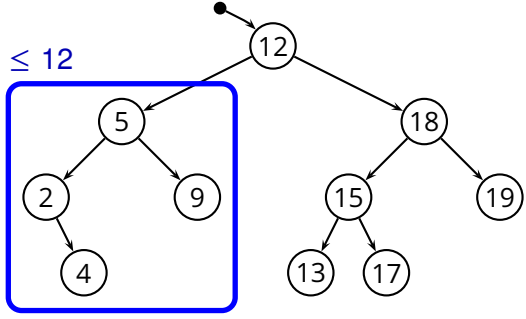

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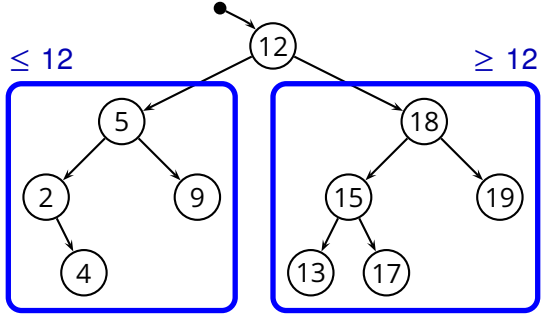

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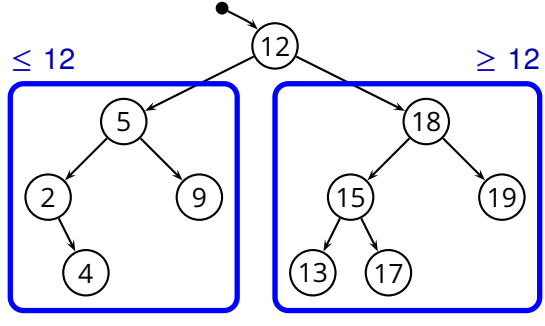

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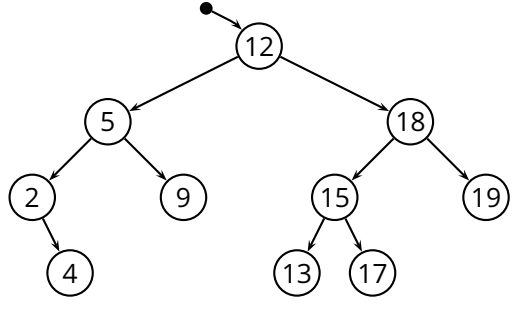

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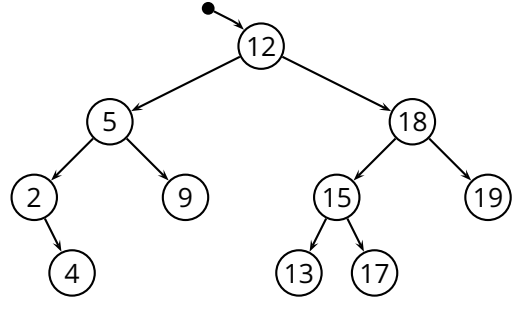

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And then we need a “starter” procedure

```
Inorder-Tree-Walk-Start(T)
1  Inorder-Tree-Walk(T.root)
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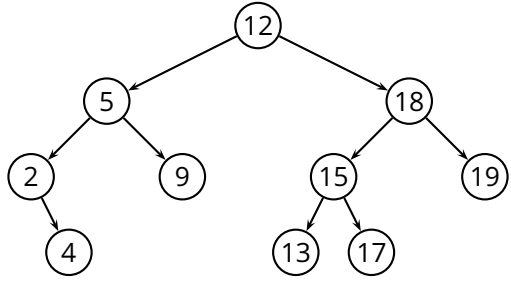

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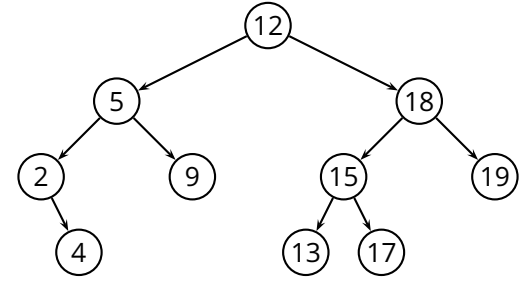

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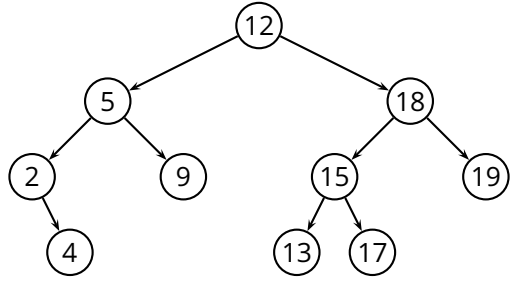

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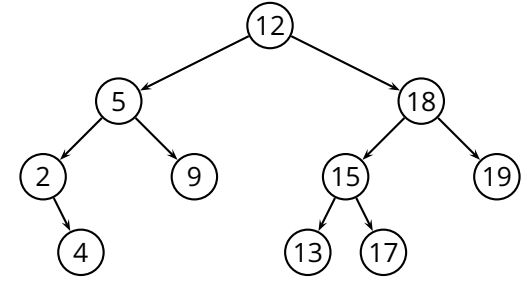

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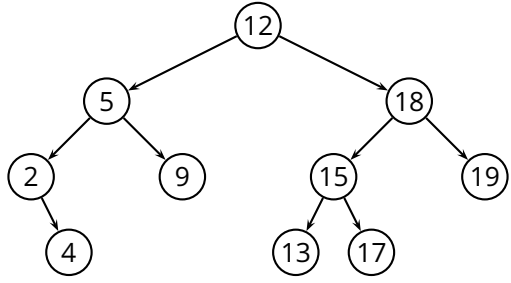

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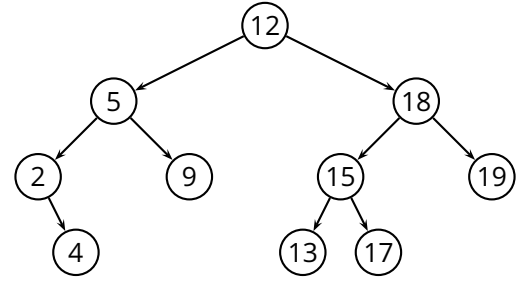

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- Can we do better? No!
  - the length of the output is $\Theta(n)$

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- So, the minimum key is in all the way to the left
  - similarly, the maximum key is all the way to the right

Tree-Minimum(x)

```
1  while x.left ≠ nil
2        x = x.left
3  return x
```

Tree-Maximum(x)

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1  while x.right ≠ nil
2        x = x.right
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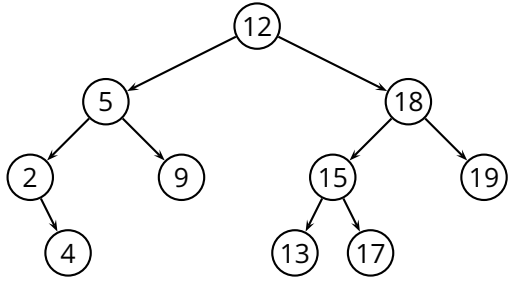

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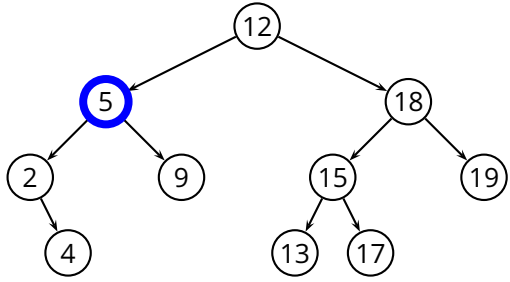

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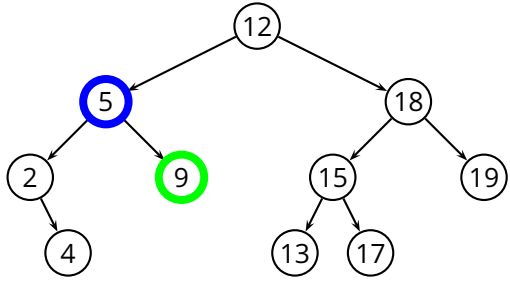

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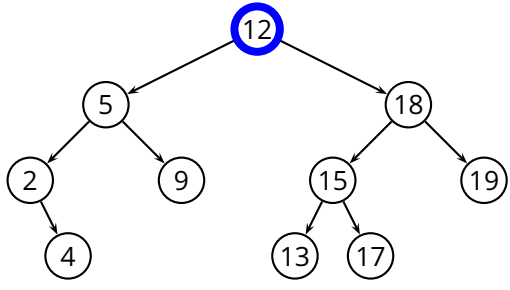

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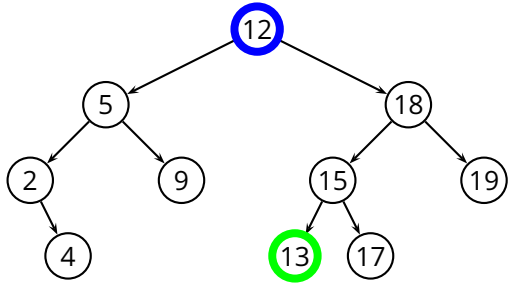

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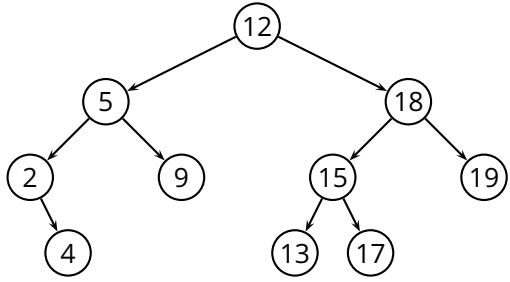

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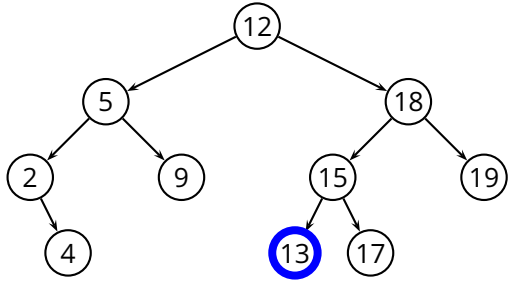

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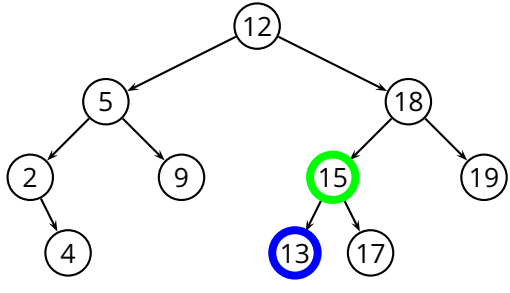

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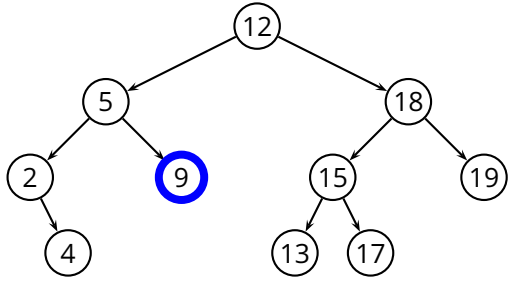

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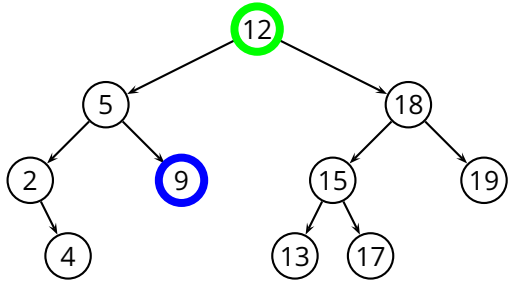

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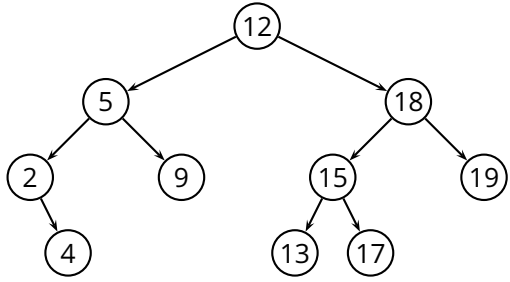

- The successor of x is the *minimum* of the *right* subtree of x, if that exists
- Otherwise it is the *first ancestor* a of x such that x falls in the *left* subtree of a

Successor and Predecessor(2)

```
Tree-Successor(x)
1  if x.right ≠ nil
2        return Tree-Minimum(x.right)
3  y = x.parent
4  while y ≠ nil and x = y.right
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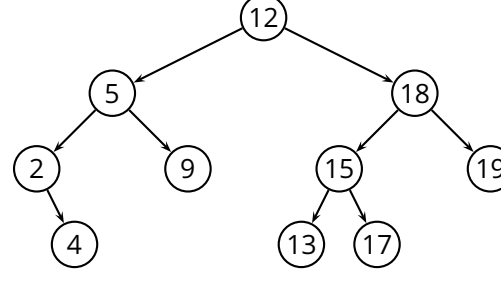

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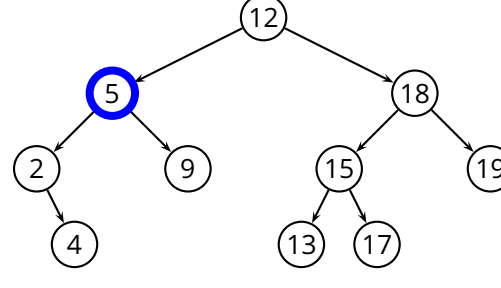

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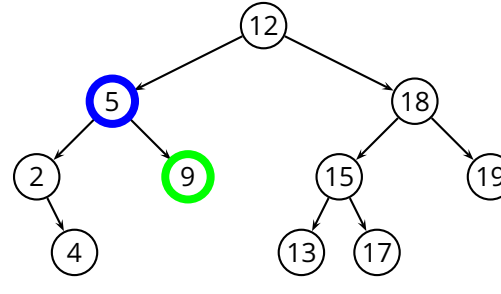

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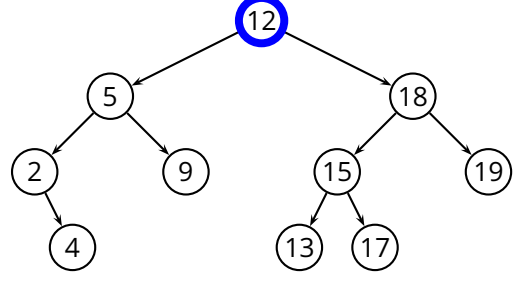

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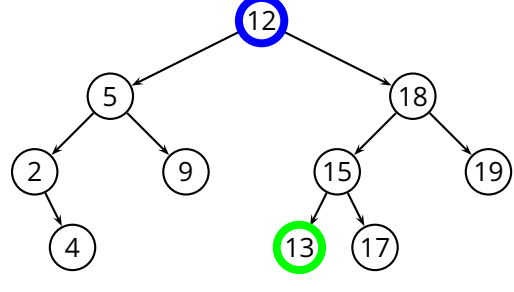

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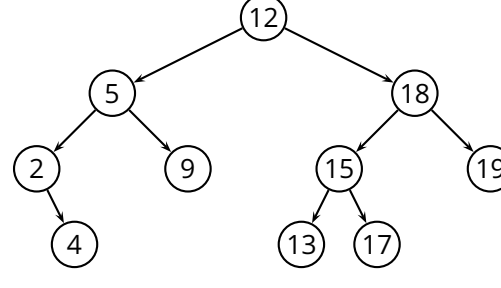

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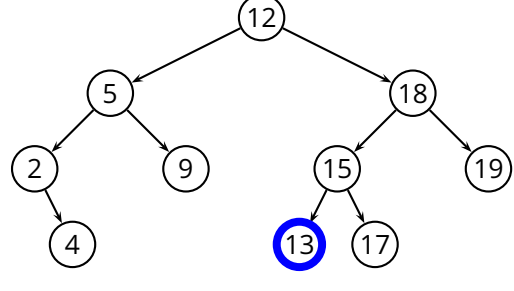

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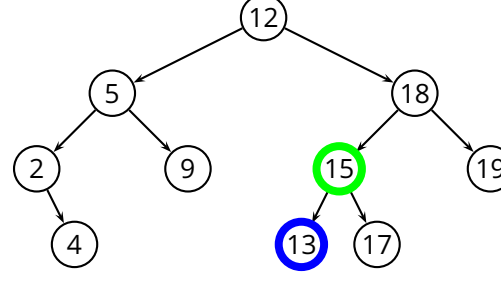
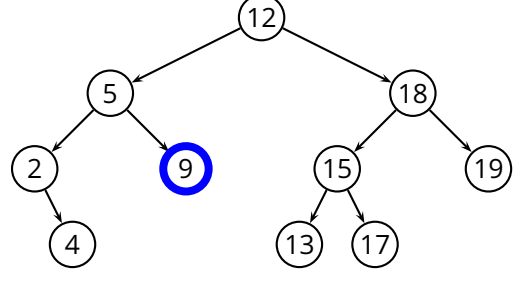
6        y = y.parent
7  return y
```



Successor and Predecessor(2)

```
Tree-Successor(x)
1  if x.right ≠ nil
2        return Tree-Minimum(x.right)
3  y = x.parent
4  while y ≠ nil and x = y.right
5        x = y
6        y = y.parent
7  return y
```



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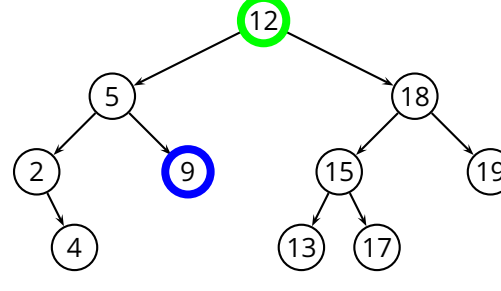

Search

- *Binary search* (thus the name of the tree)

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```
Tree-Search(x, k)
1  if x = nil or k = x.key
2        return x
3  if k < x.key
4        return Tree-Search(x.left, k)
5  else return Tree-Search(x.right, k)
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$$T(n) = O(n)$$

Search (2)

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- Iterative *binary search*

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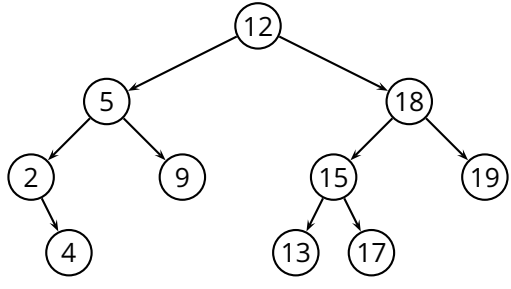
```
Iterative-Tree-Search(T, k)
1  x = T.root
2  while x ≠ nil ∧ k ≠ x.key
3        if k < x.key
4              x = x.left
5        else x = x.right
6  return x
```

Insertion

Insertion



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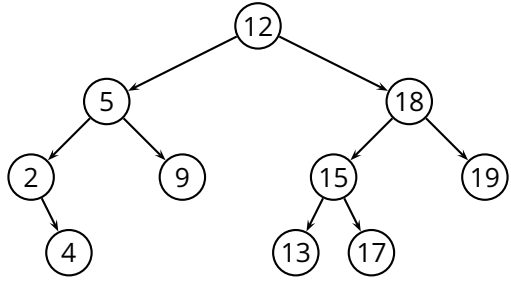

- *Idea*
  - in order to insert x, we *search* for x (more precisely $x.key$)
  - if we don't find it, we add it where the search stopped

Insertion (2)

```
Tree-Insert(T, z)
 1  y = nil
 2  x = T.root
 3  while x ≠ nil
 4        y = x
 5        if z.key < x.key
 6              x = x.left
 7        else x = x.right
 8  z.parent = y
 9  if y = nil
10        T.root = z
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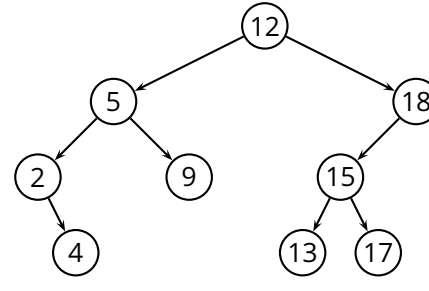

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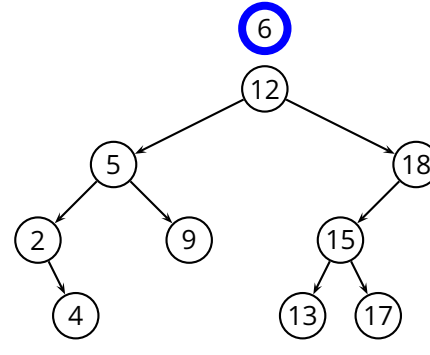

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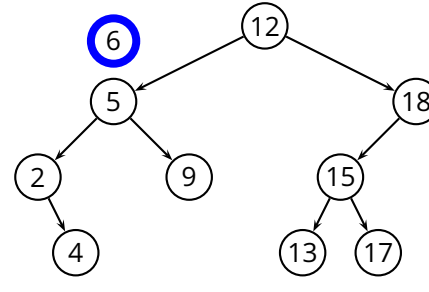

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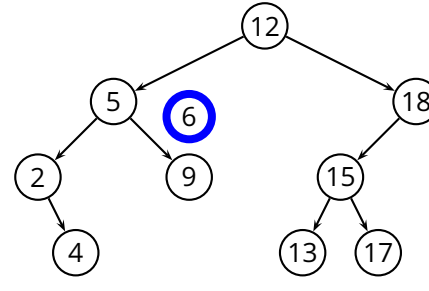

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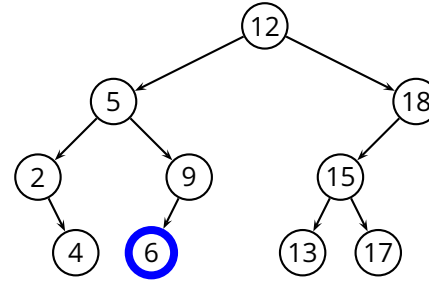

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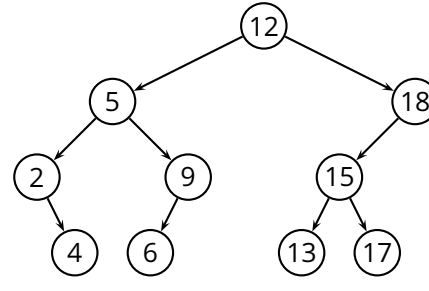

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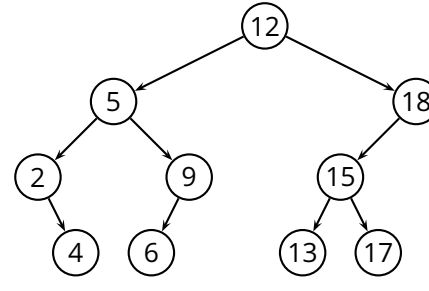
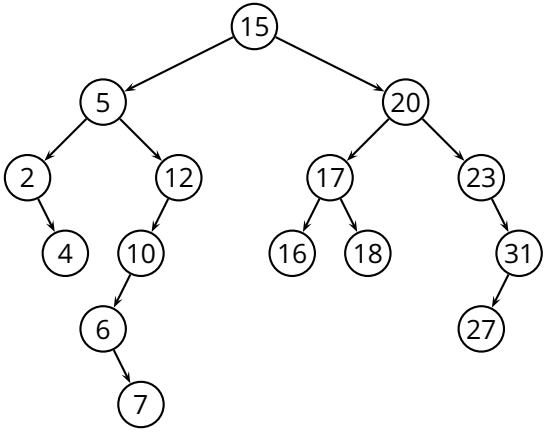


$T(n) = \Theta(h)$

Deletion

Deletion



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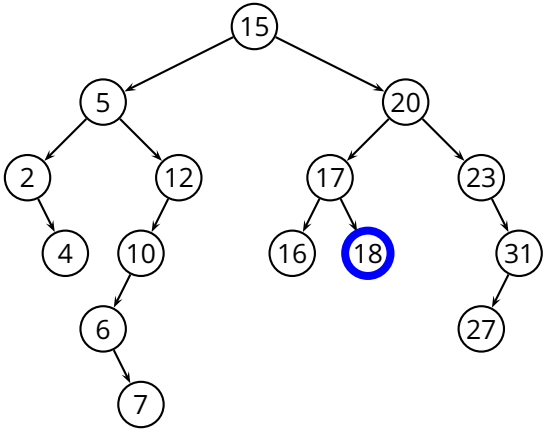
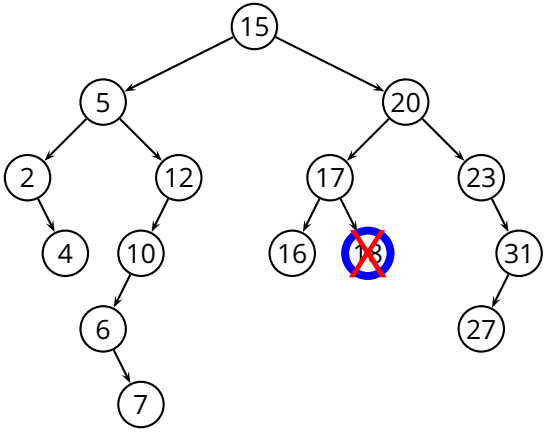


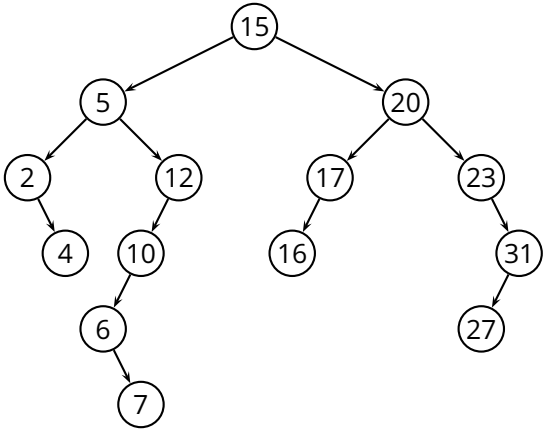
1. z has no children

Deletion



1. z has no children
   - simply remove z

Deletion



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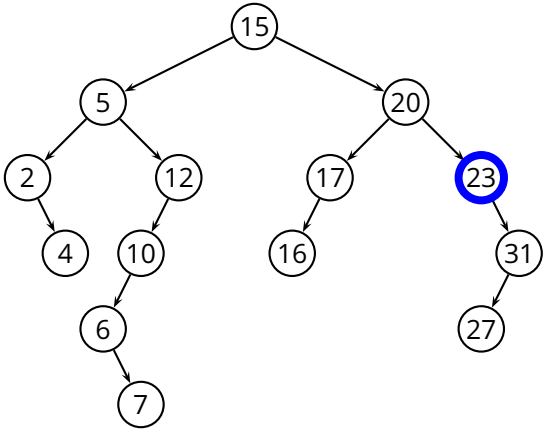

1. z has no children
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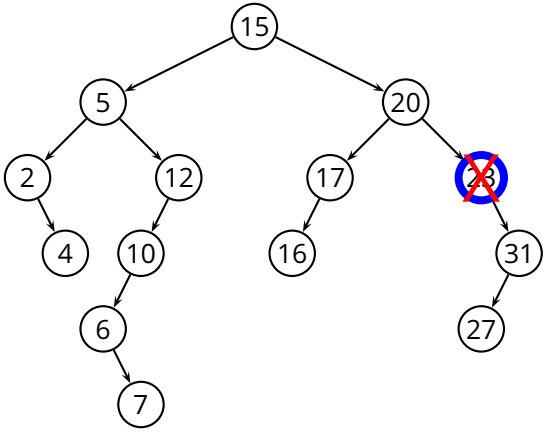
2. z has one child

Deletion



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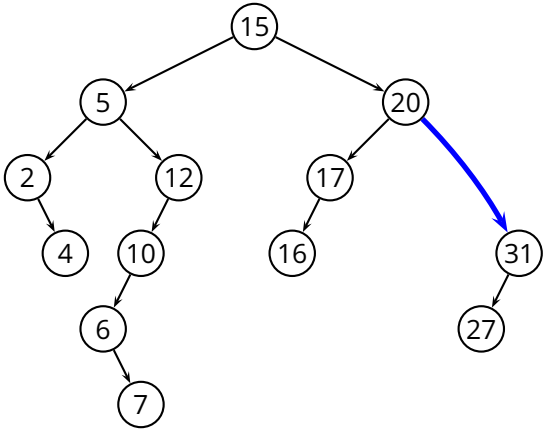

1. z has no children
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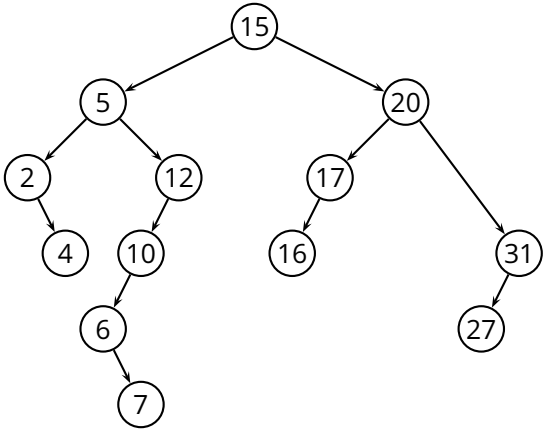
2. z has one child
   - remove z
   - connect $z.parent$ to $z.right$

Deletion



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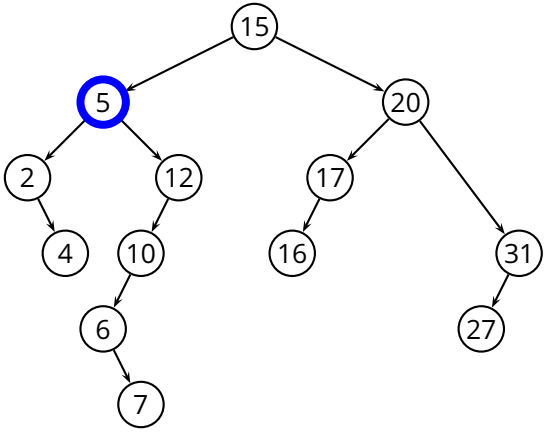
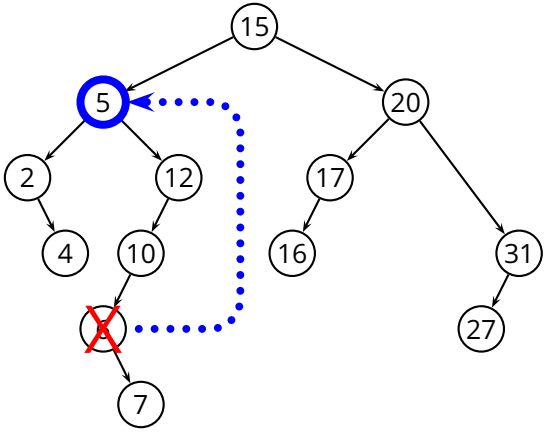


1. z has no children
   - simply remove z
2. z has one child
   - remove z
   - connect $z.parent$ to $z.right$
3. z has two children

Deletion



1. z has no children
   - simply remove z
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   - connect $z.\mathit{parent}$ to $z.\mathit{right}$
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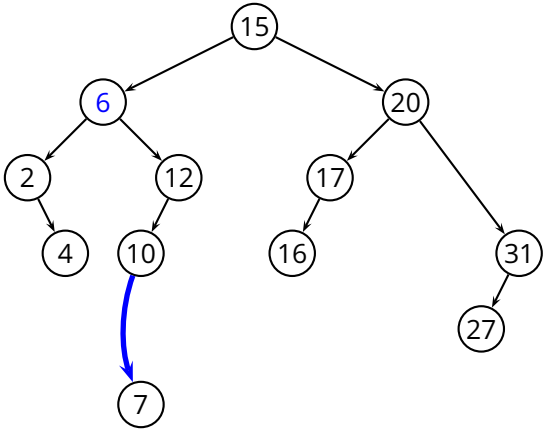
   - replace z with $y = \textbf{Tree-Successor}(z)$
   - remove y (1 child!)

Deletion



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   - connect $y.parent$ to $y.right$

Deletion (2)

```
Tree-Delete(T, z)
 1  if z.left = nil or z.right = nil
 2        y = z
 3  else y = Tree-Successor(z)
 4  if y.left ≠ nil
 5        x = y.left
 6  else x = y.right
 7  if x ≠ nil
 8        x.parent = y.parent
 9  if y.parent == nil
10        T.root = x
11  else if y = y.parent.left
12              y.parent.left = x
13        else y.parent.right = x
14  if y ≠ z
15        z.key = y.key
16        copy any other data from y into z
```

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- Insertion, search, and deletion operations have complexity $\Theta(h)$, where h is the height of the tree

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- $h = O(n)$ in some particular cases
  - i.e., with ordered sequences
  - the problem is that the "worst" case is not that uncommon
- ***Idea:*** use randomization to turn all cases into the average case

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- ***Idea 2:*** we can obtain a random permutation of the input sequence by randomly alternating two insertion procedures
  - *tail insertion:* this is what **Tree-Insert** does
  - *head insertion:* for this we need a new procedure **Tree-Root-Insert**
    - inserts n in T as if n was inserted as the first element

Randomized Insertion (2)

```
Tree-Randomized-Insert1(T, z)
1  r = uniformly random value from {1, ..., t.size + 1}
2  if r = 1
3        Tree-Root-Insert(T, z)
4  else Tree-Insert(T, z)
```

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- It is true that any node has the same probability of being inserted at the top
  - this suggests a recursive application of this same procedure

Randomized Insertion (3)

Randomized Insertion (3)

```
Tree-Randomized-Insert(t, z)
 1  if t = nil
 2       return z
 3  r = uniformly random value from {1, ..., t.size + 1}
 4  if r = 1                          // Pr[r = 1] = 1/(t.size + 1)
 5       z.size = t.size + 1
 6       return Tree-Root-Insert(t, z)
 7  if z.key < t.key
 8       t.left = Tree-Randomized-Insert(t.left, z)
 9  else t.right = Tree-Randomized-Insert(t.right, z)
10  t.size = t.size + 1
11  return t
```

Randomized Insertion (3)

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Tree-Randomized-Insert(t, z)
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 5       z.size = t.size + 1
 6       return Tree-Root-Insert(t, z)
 7  if z.key < t.key
 8       t.left = Tree-Randomized-Insert(t.left, z)
 9  else t.right = Tree-Randomized-Insert(t.right, z)
10  t.size = t.size + 1
11  return t
```
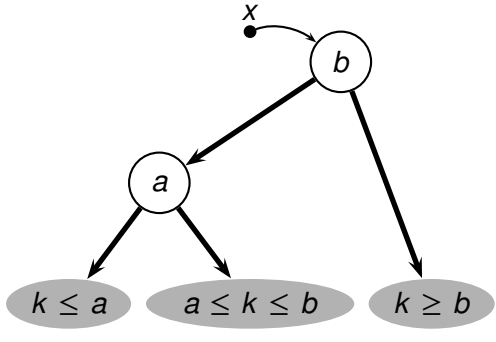
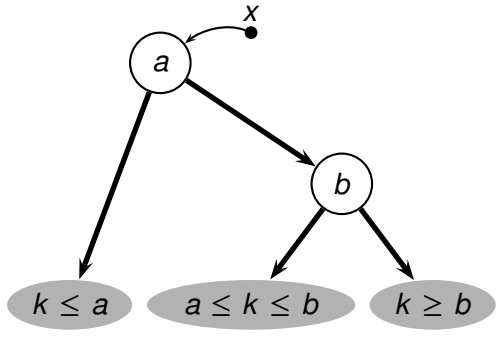

- Looks like this one really simulates a random permutation...

Rotation

Rotation

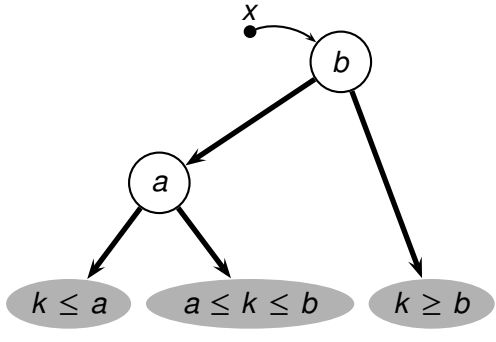


Rotation



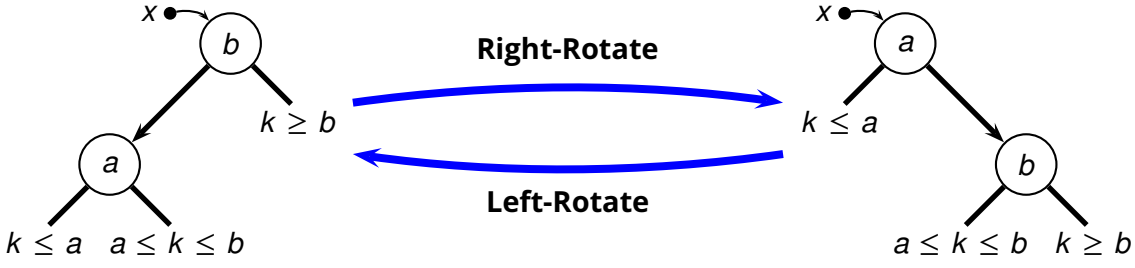
- $x =$ **Right-Rotate**(x)

Rotation



- $x =$ **Right-Rotate**(x)
- $x =$ **Left-Rotate**(x)

Rotation



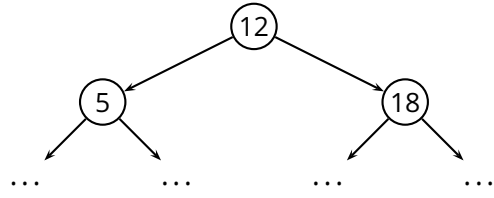
Right-Rotate(x)

```
1  l = x.left
2  x.left = l.right
3  l.right = x
4  return l
```

Left-Rotate(x)

```
1  r = x.right
2  x.right = r.left
3  r.left = x
4  return r
```

Root Insertion



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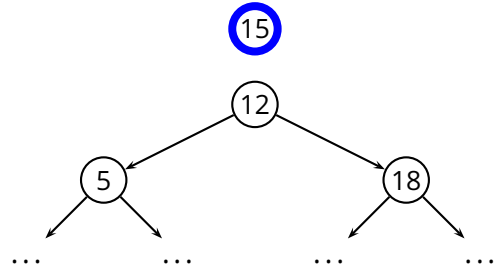
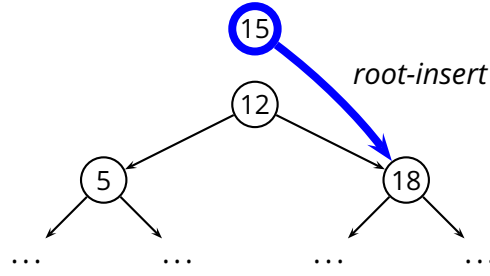
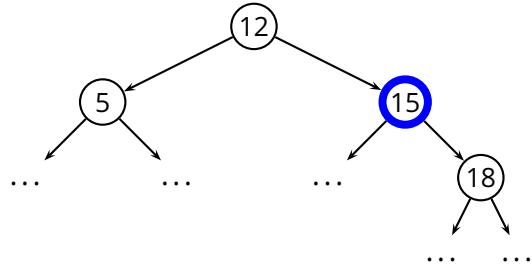


Root Insertion



1. Recursively insert z at the root of the appropriate subtree (right)

Root Insertion



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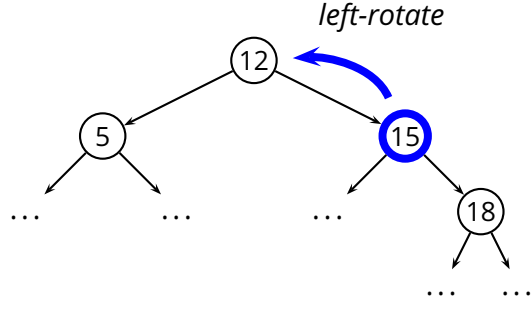
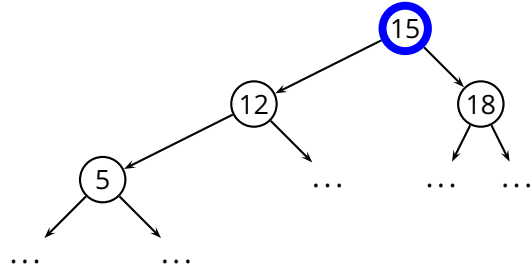


1. Recursively insert *z* at the root of the appropriate subtree (right)

2. Rotate *x* with *z* (left-rotate)

Root Insertion



1. Recursively insert z at the root of the appropriate subtree (right)

2. Rotate x with z (left-rotate)

Root Insertion (2)

```
Tree-Root-Insert(x, z)
1  if x = nil
2        return z
3  if z.key < x.key
4        x.left = Tree-Root-Insert(x.left, z)
5        return Right-Rotate(x)
6  else x.right = Tree-Root-Insert(x.right, z)
7        return Left-Rotate(x)
```

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    - relatively expensive but “amortized” operations
  - *optimized data structures:* a self-balanced data structure
    - guaranteed $O(\log n)$ complexity bounds