# More on Sorting: Quick Sort and Heap Sort

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March 12, 2020

# Outline

- Another divide-and-conquer sorting algorithm
- The *heap*
- Heap sort

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# Using the Partitioning Algorithm

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- *Can we partition A* ***in place****?*

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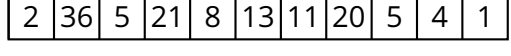


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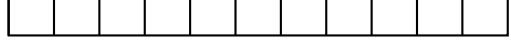

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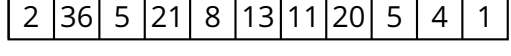


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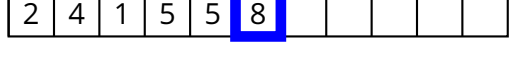

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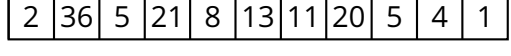


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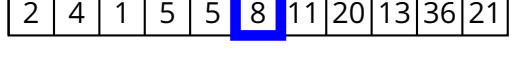

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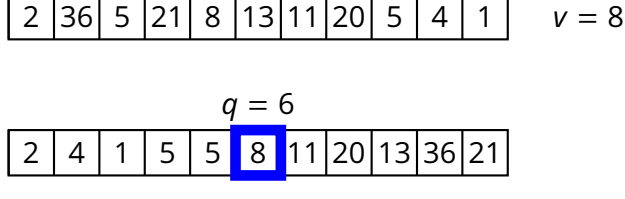

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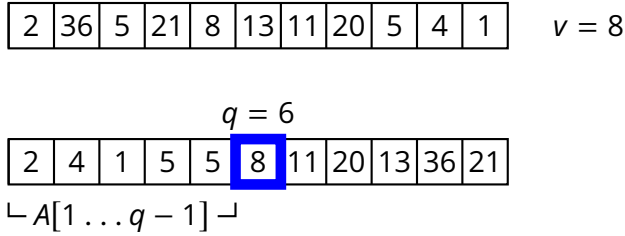

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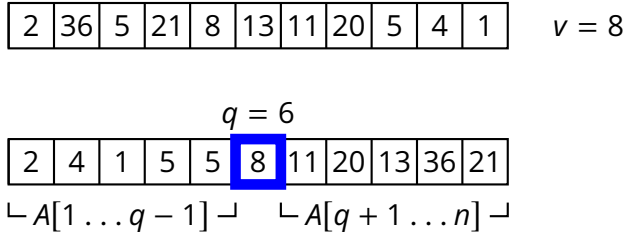

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- ***Divide:*** partition $A$ in $A[1 \ldots q-1]$ and $A[q+1 \ldots n]$ such that

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```
QUICKSORT(A, begin, end)
1  if begin < end
2        q = PARTITION(A, begin, end)
3        QUICKSORT(A, begin, q − 1)
4        QUICKSORT(A, q + 1, end)
```

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  - i.e., *assume all elements are greater than the pivot*
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- *Loop invariant*
  - $begin \leq k < q \Longrightarrow A[k] \leq v$
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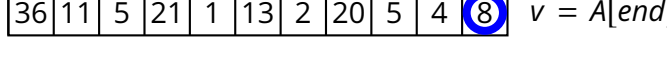

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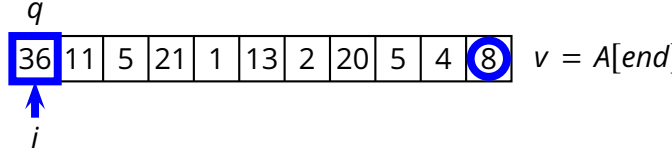

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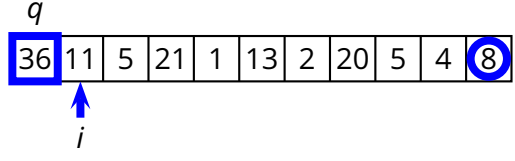

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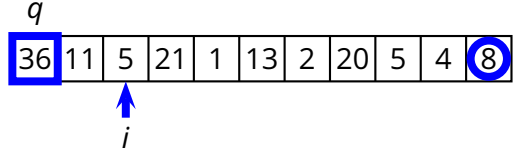

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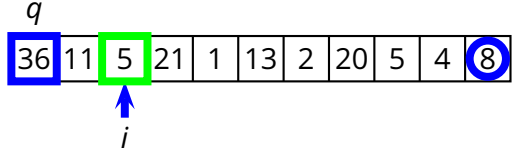

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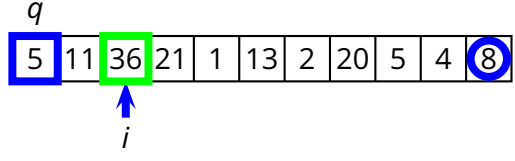

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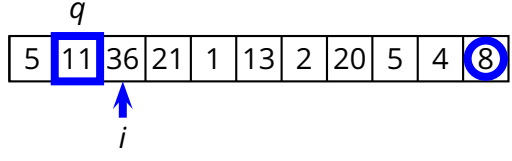

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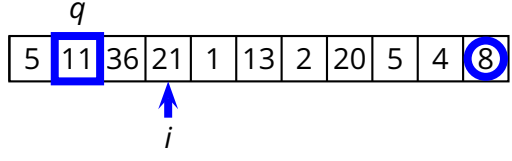

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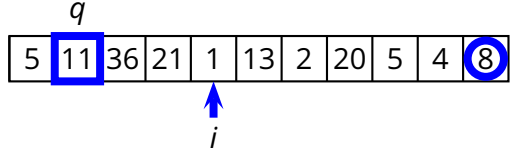

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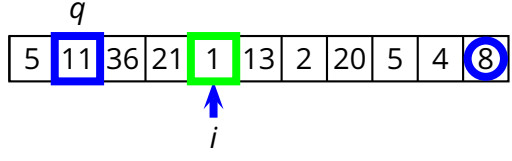

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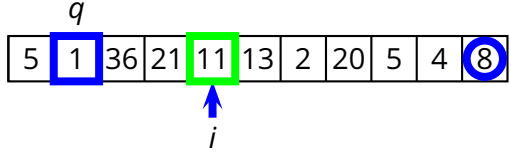

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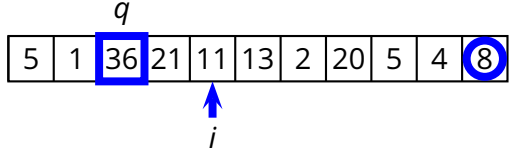

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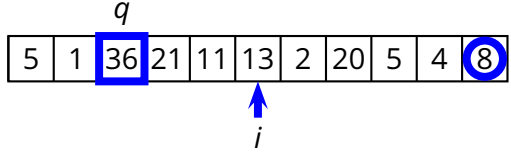

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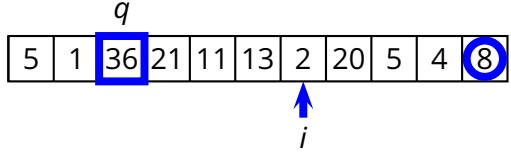

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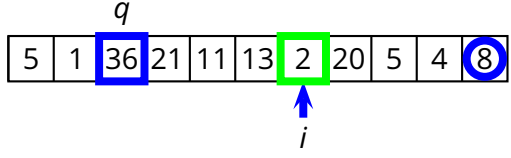

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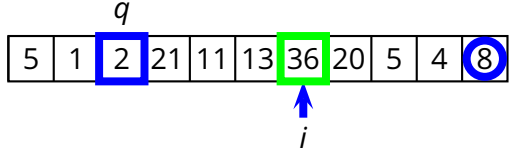

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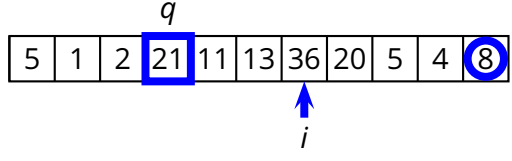

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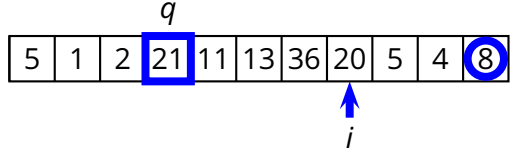

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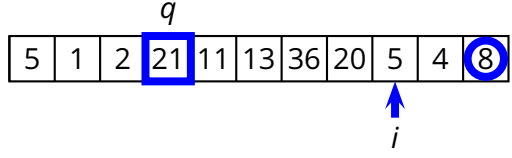

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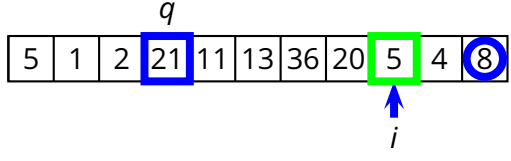

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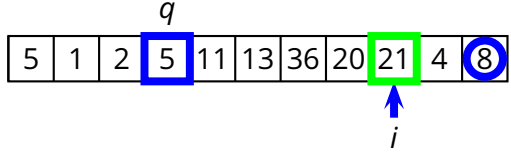

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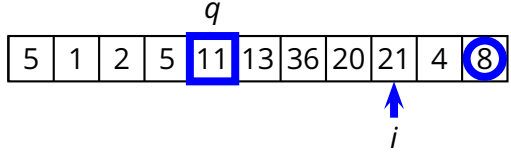

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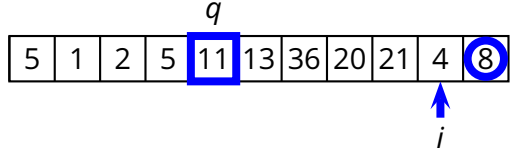

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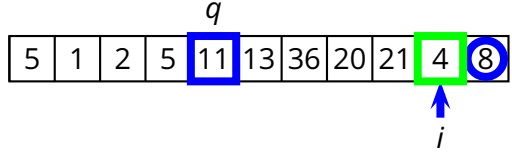

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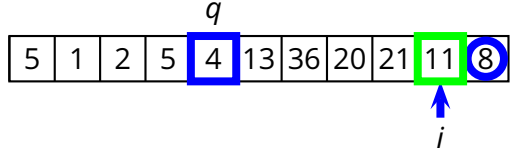

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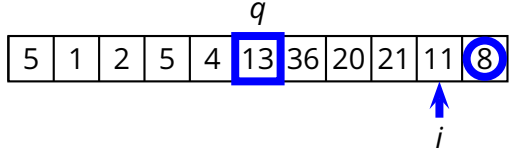

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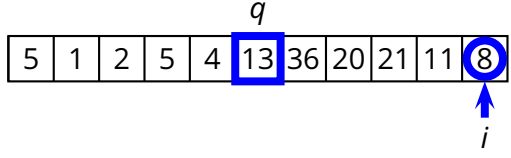

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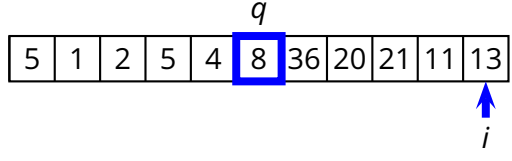

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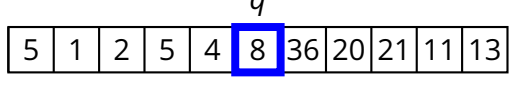

# Complete QUICKSORT Algorithm

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PARTITION(A, begin, end)
1  q = begin
2  v = A[end]
3  for i = begin to end
4       if A[i] ≤ v
5            swap A[i] and A[q]
6            q = q + 1
7  return q − 1
```

```
QUICKSORT(A, begin, end)
1  if begin < end
2       q = PARTITION(A, begin, end)
3       QUICKSORT(A, begin, q − 1)
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$$T(n) = \Theta(n)$$

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$$T(n) = \Theta(n^2)$$

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  - $q = \lceil n/2 \rceil$

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  - $q = \lceil n/2 \rceil$
  - the partition transforms $P$ of size $n$ into *two* problems $P$ of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil - 1$, respectively

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$$T(n) = \Theta(n \log n)$$

# Sorting Algorithms Seen So Far

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|---|---|---|---|---|
| | *worst* | *average* | *best* | |
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| **SELECTION-SORT** | $\Theta(n^2)$ | $\Theta(n^2)$ | $\Theta(n^2)$ | yes |
| **BUBBLE-SORT** | $\Theta(n^2)$ | $\Theta(n^2)$ | $\Theta(n^2)$ | yes |
| **MERGE-SORT** | $\Theta(n \log n)$ | $\Theta(n \log n)$ | $\Theta(n \log n)$ | no |

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| Algorithm | Complexity | | | In place? |
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# Binary Heap

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- Our first real ***data structure***

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# Binary Heap: Structure

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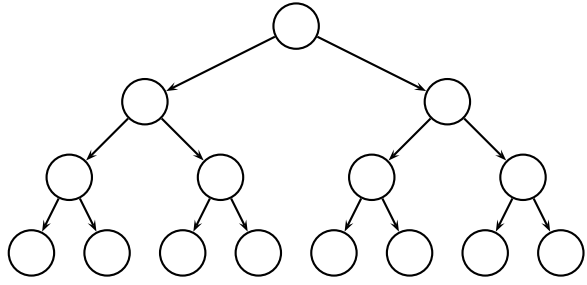

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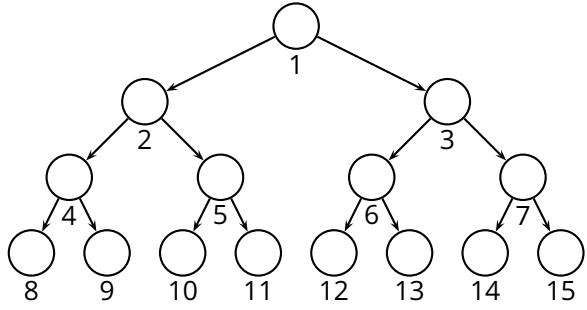

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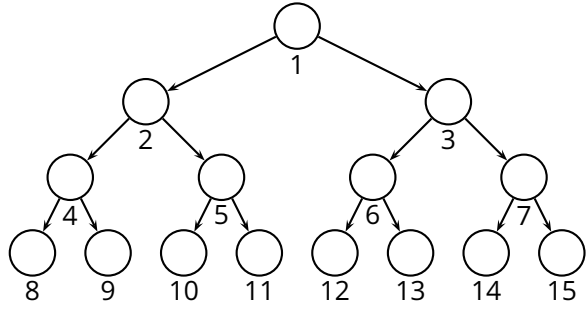


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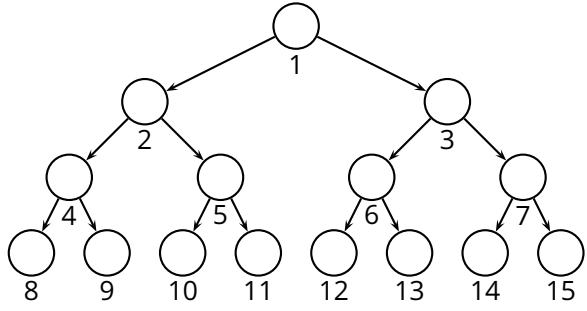


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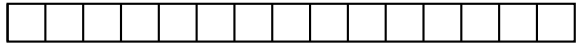

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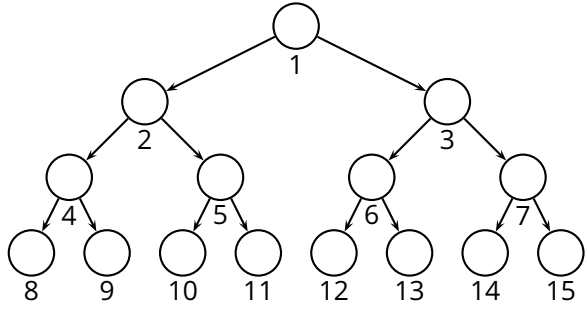


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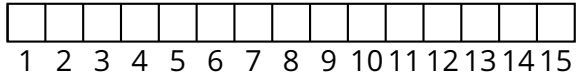

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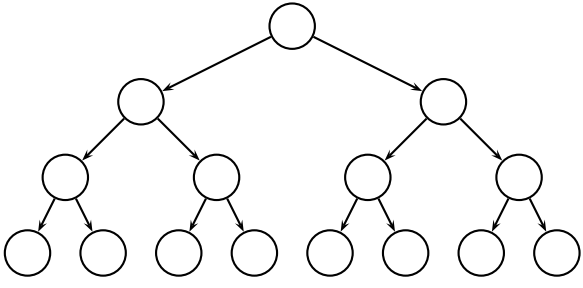

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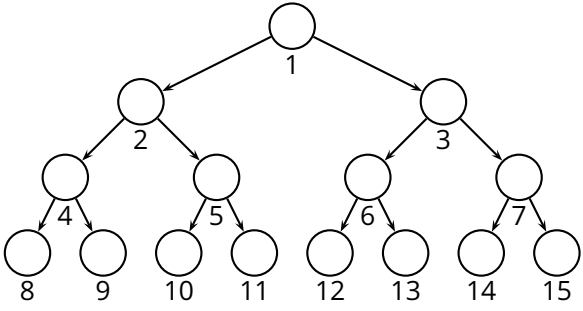

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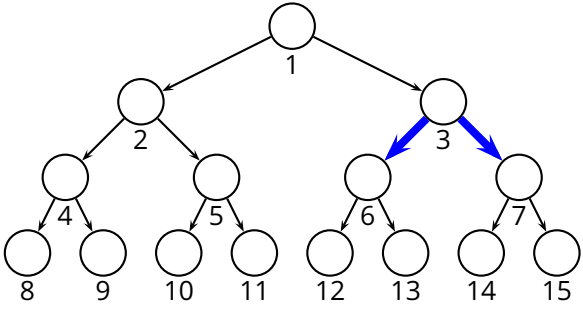

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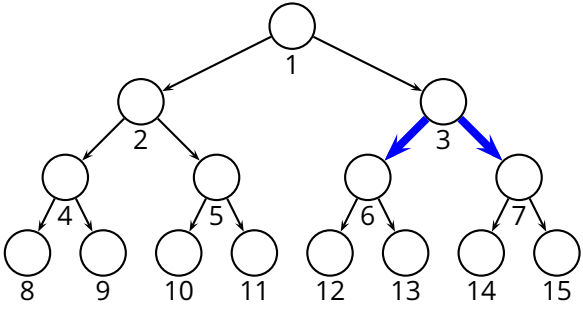


**PARENT**$(i)$
    **return** $\lfloor i/2 \rfloor$
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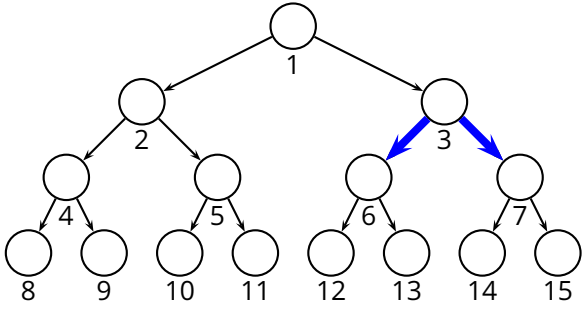


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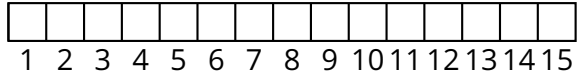

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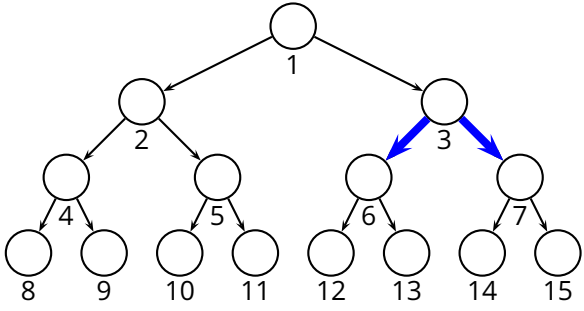


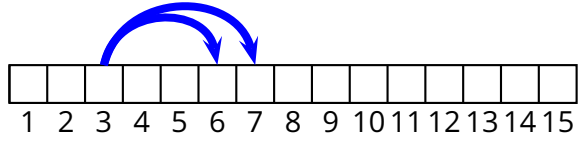


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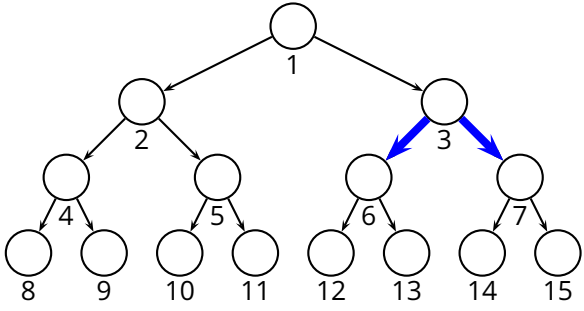


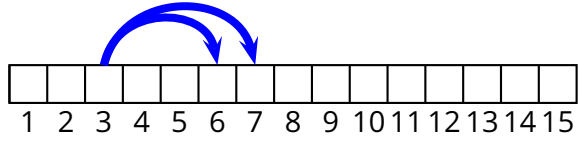


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- ***Max-heap property:*** for all $i > 1$ $\boxed{A[\textbf{PARENT}(i)] \geq A[i]}$

# Example

- *Max-heap property:* for all $i > 1$ $A[\textsc{Parent}(i)] \geq A[i]$

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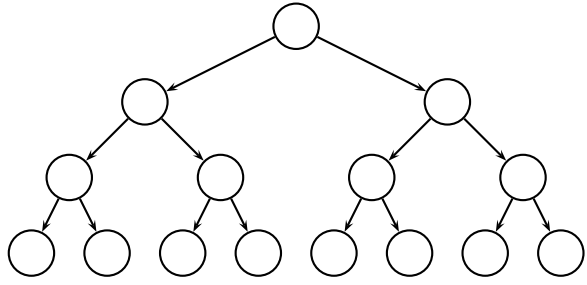

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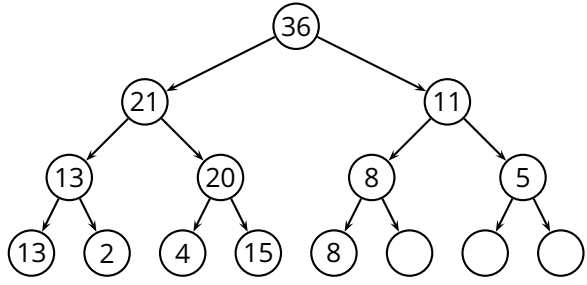

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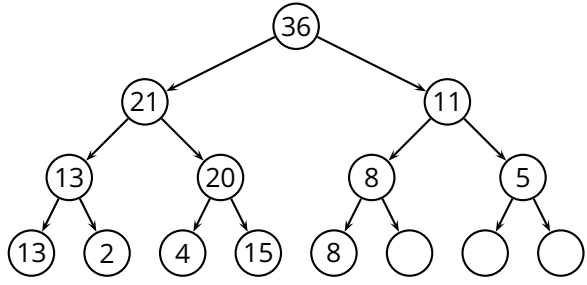


- Where is the max element?

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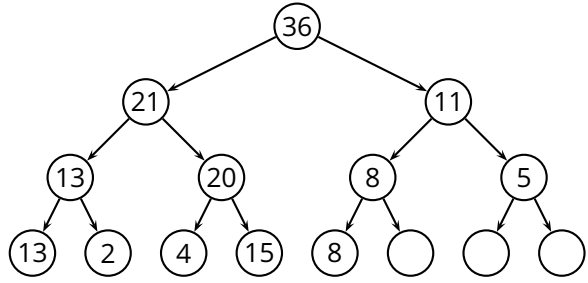


- Where is the max element?
- How can we implement **HEAP-EXTRACT-MAX**?

# Heap-Extract-Max

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  - extract the max key
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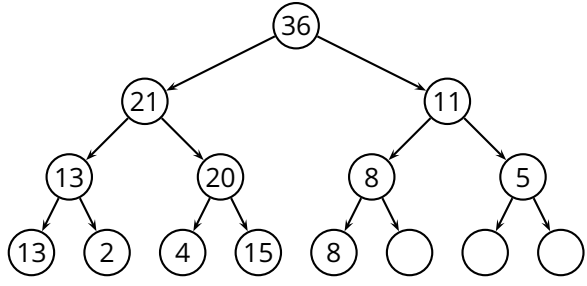

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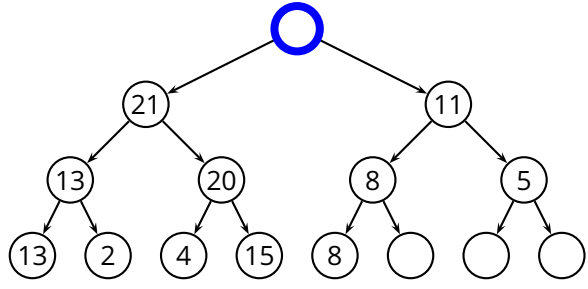

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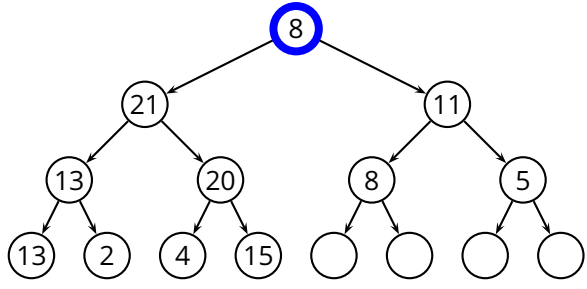

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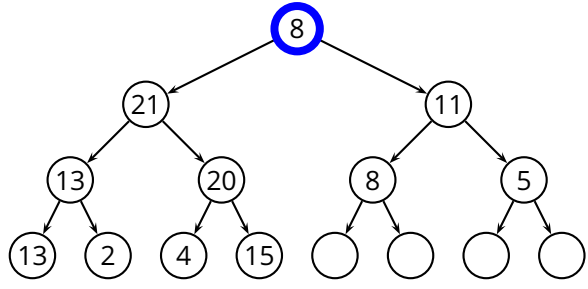


- Now we have two subtrees where the *max-heap property* holds

- **MAX-HEAPIFY**$(A, i)$ procedure
  - *assume:* the *max-heap property* holds in the subtrees of node $i$
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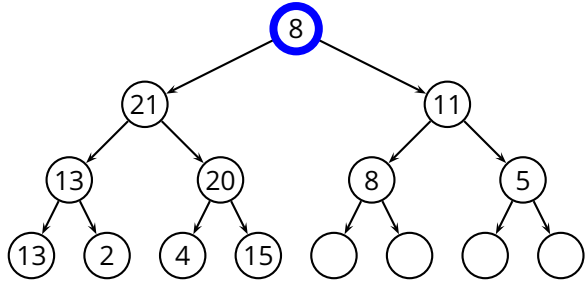

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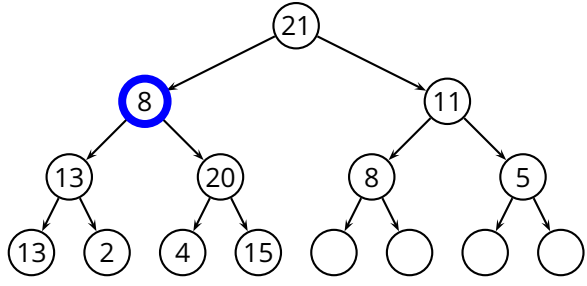

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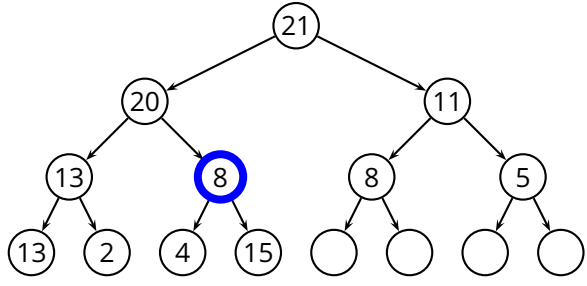

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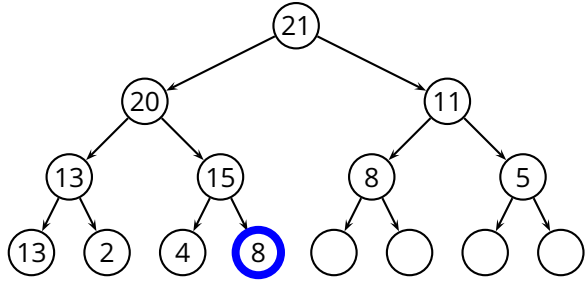

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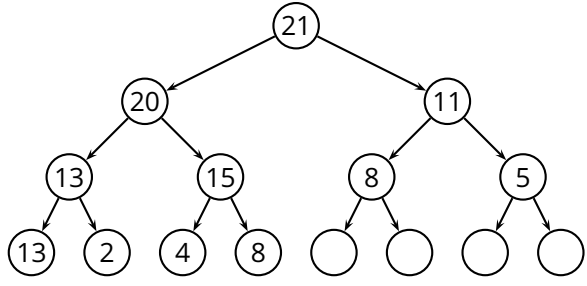

# Max-Heapify

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MAX-HEAPIFY(A, i)
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 3  if l ≤ A.heap-size and A[l] > A[i]
 4        largest = l
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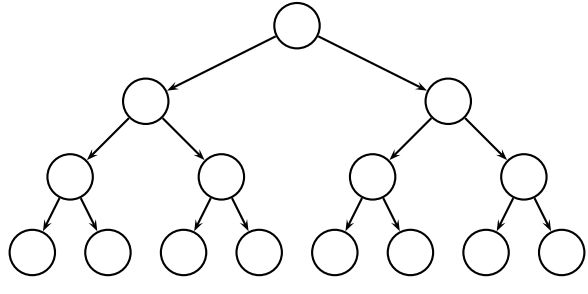

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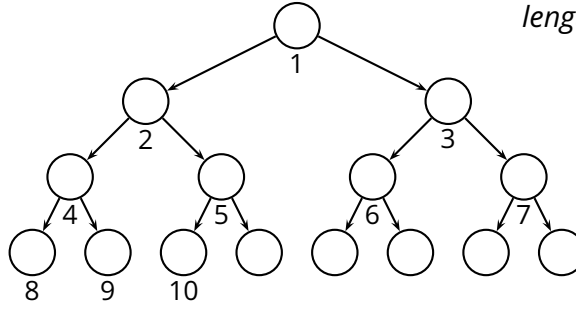


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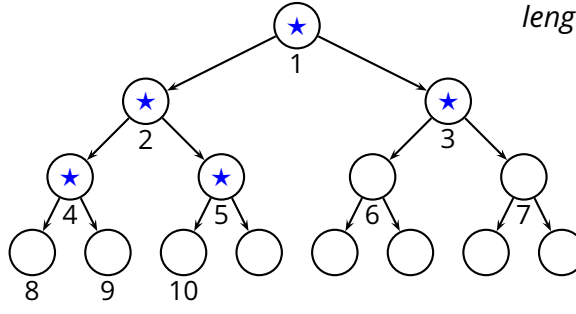


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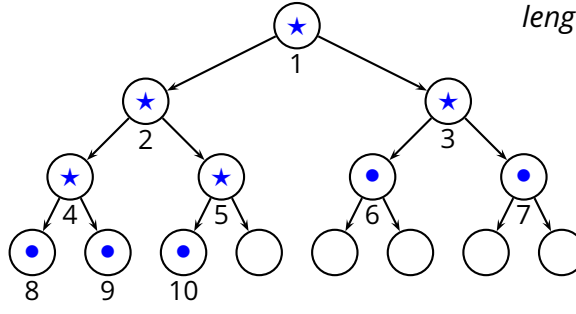


$length(A) = 10$

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BUILD-MAX-HEAP(A)
1  A.heap-size = length(A)
2  for i = ⌊length(A)/2⌋ downto 1
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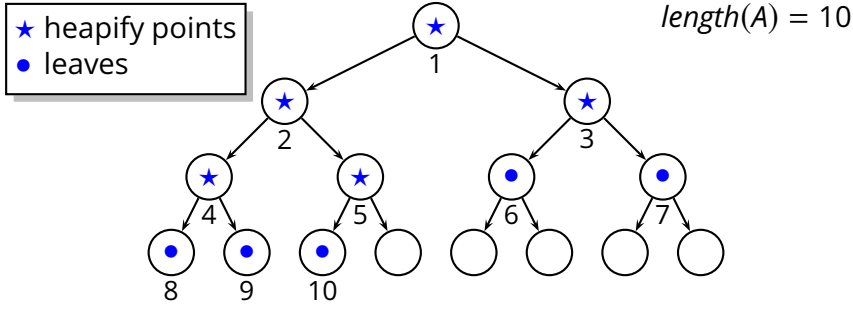

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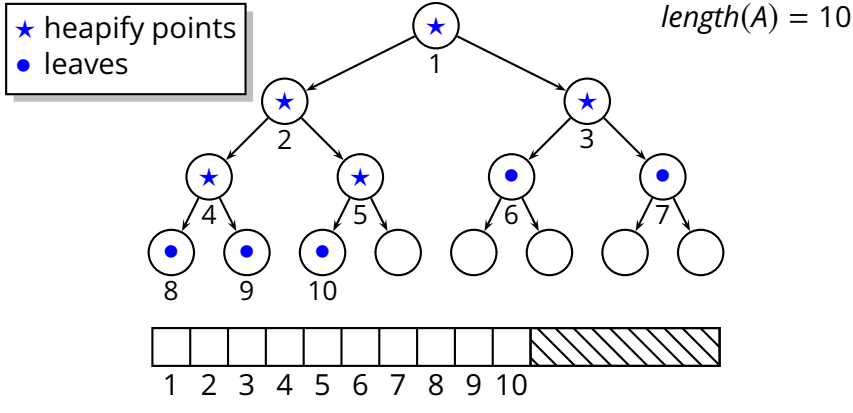

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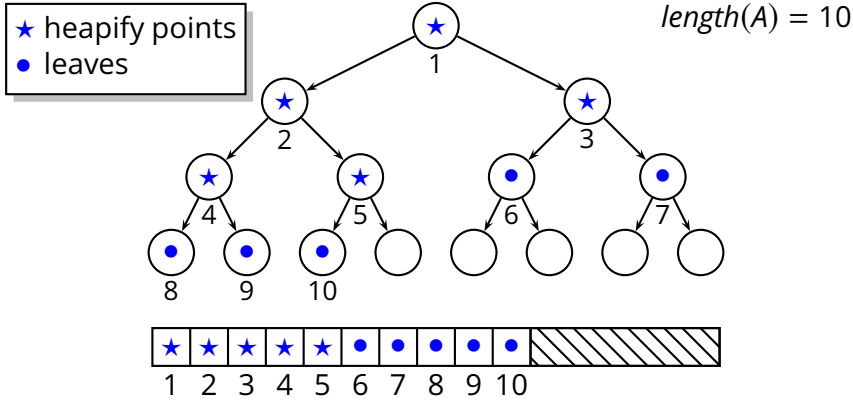

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- Benefits
  - in-place sorting; worst-case is $\Theta(n \log n)$

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