# Analysis of Insertion Sort

Antonio Carzaniga

Faculty of Informatics
Università della Svizzera italiana

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# Outline

- Sorting
- Insertion Sort
- Analysis

- **Input:** a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$

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|---|---|---|---|---|---|---|---|---|---|
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# Sorting

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- Typically, $A$ is implemented as an array

***in-place sort***

$$A = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 2 & 3 & 4 & 5 & 6 & 6 & 7 & 8 & 9 & 11 \\ \hline \end{array}$$

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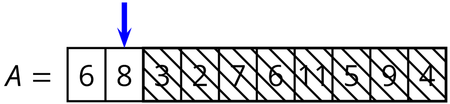

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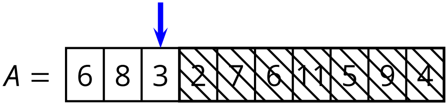

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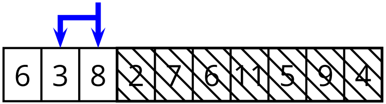
$A =$ 

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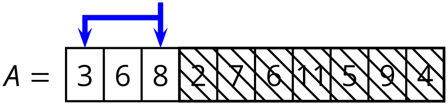

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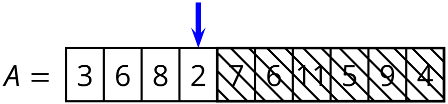

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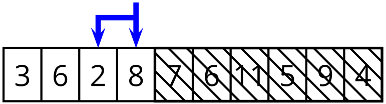

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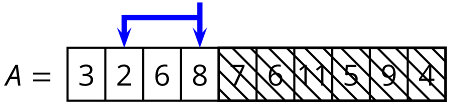

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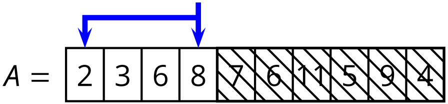

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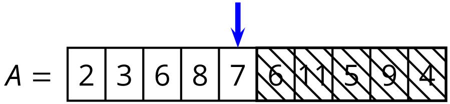

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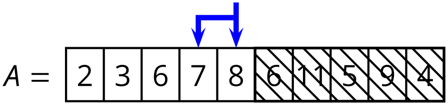

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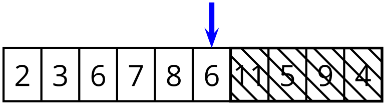

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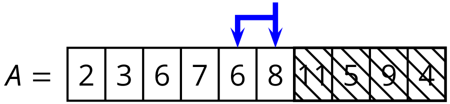

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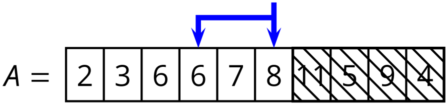

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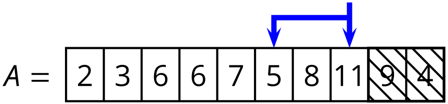

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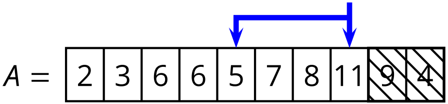

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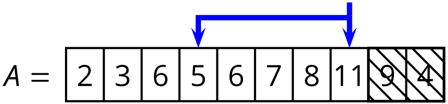

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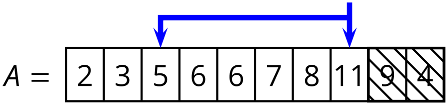

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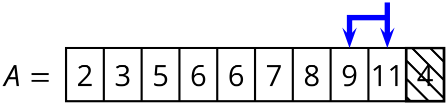

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$$A = \boxed{2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 6 \mid 7 \mid 8 \mid 9 \mid 11}$$

# Insertion Sort (2)

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
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- Is **INSERTION-SORT** *correct?*
- What is the time complexity of **INSERTION-SORT**?
- Can we do better?

# Complexity of INSERTION-SORT

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# Complexity of INSERTION-SORT

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INSERTION-SORT(A)
1  for i = 2 to length(A)
2      j = i
3      while j > 1 and A[j − 1] > A[j]
4          swap A[j] and A[j − 1]
5          j = j − 1
```

- Outer loop (lines 1–5) runs exactly $n - 1$ times (with $n = \mathit{length}(A)$)
- What about the inner loop (lines 3–5)?
  - best, worst, and average case?

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- **Best case:**

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- **Worst case:**

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- **Best case:** the inner loop is *never* executed
  - what case is this?

- **Worst case:** the inner loop is executed exactly $j - 1$ times for every iteration of the outer loop
  - what case is this?

- The worst-case complexity is when the inner loop is executed exactly $j - 1$ times, so

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- We want ***a formal proof of correctness***
  - does not seem straightforward...

# The Logic of Algorithmic Steps

**Example 1:** (straight-line program)

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BIGGER(n)
// must return a value greater than n
m = n * n + 1
return m
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**Example 2:** (branching)

```
SORTTWO(A)
// must sort (in-place) an array of 2 elements
if A[1] > A[2]
    t = A[1]
    A[1] = A[2]
    A[2] = t
```

# Loop Invariants

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- Then, we only need to prove that the algorithm terminates

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  - *the invariant must reflect the structure of the algorithm*
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- Proof of validity (i.e., that *C* is indeed a loop invariant): typical *proof by induction*
  - ***initialization:*** we must prove that
    *the invariant C is true before entering the loop*
  - ***maintenance:*** we must prove that
    ***if*** *C is true at the beginning of a cycle* ***then*** *it remains true after one cycle*

# Loop Invariant for INSERTION-SORT

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
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- The main idea is to insert $A[i]$ in $A[1 \mathinner{.\,.} i-1]$ so as to maintain a *sorted subsequence* $A[1 \mathinner{.\,.} i]$

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- ***Invariant:*** (outer loop) *the subarray* $A[1 \,..\, i-1]$ *consists of the elements originally in* $A[1 \,..\, i-1]$ *in sorted order*

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- **Initialization:** $j = 2$, so $A[1\,..\,j-1]$ is the single element $A[1]$
  - $A[1]$ contains the original element in $A[1]$
  - $A[1]$ is trivially sorted

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- **Maintenance:** informally, if $A[1 \,..\, i-1]$ is a permutation of the original $A[1 \,..\, i-1]$ and $A[1 \,..\, i-1]$ is sorted (invariant), then *if* we enter the inner loop:
  - shifts the subarray $A[k \,..\, i-1]$ by one position to the right
  - inserts *key*, which was originally in $A[i]$ at its proper position $1 \leq k \leq i-1$, in sorted order

# Loop Invariant for INSERTION-SORT (4)

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
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- **Termination:** INSERTION-SORT terminates with $i = length(A) + 1$; the invariant states that
  - $A[1 \,..\, i-1]$ is a permutation of the original $A[1 \ldots i-1]$
  - $A[1 \,..\, i-1]$ is sorted

  Given the termination condition, $A[1 \,..\, i-1]$ is the whole $A$

  So INSERTION-SORT is *correct!*

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  - $P$ formally defines a *correctness* condition
  - assume, for simplicity, that $A$ consists of one loop

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   - prove that the loop terminates, with some exit condition $X$
5. Prove that $X \wedge C \implies P$, which means that $A$ is correct

# Exercise: Analyze Selection-Sort

```
SELECTION-SORT(A)
n = length(A)
for i = 1 to n − 1
    smallest = i
    for j = i + 1 to n
        if A[j] < A[smallest]
            smallest = j
    swap A[i] and A[smallest]
```

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- Correctness?
  - loop invariant?
- Complexity?
  - worst, best, and average case?

# Exercise: Analyze Bubblesort

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BUBBLESORT(A)
for i = 1 to length(A)
    for j = length(A) downto i + 1
        if A[j] < A[j − 1]
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# Exercise: Analyze Bubblesort

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