Minimal Spanning Trees

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Outline

- MST problem
- Generic algorithm
- Prim and Kruskal

- Given a weighted graph $G = (V, E)$
  - with "weight" function $w : E \to \mathbb{R}$

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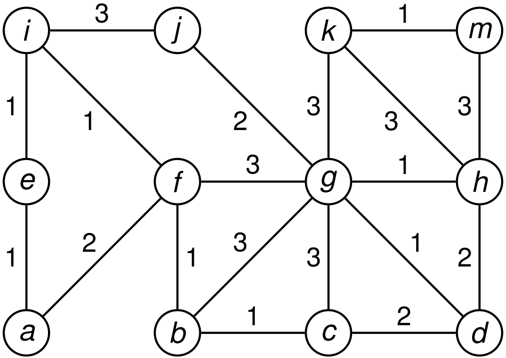
  - a ***tree***
- T "spans" the entire graph G (i.e., touches all vertexes)
  - a ***spanning tree***
- T's total weight of the tree is minimal

$$w(T) = \sum_{(u,v)\in T} w(u, v)$$

  - a ***minimum-weight spanning tree***, or "minimum spanning tree"

Example



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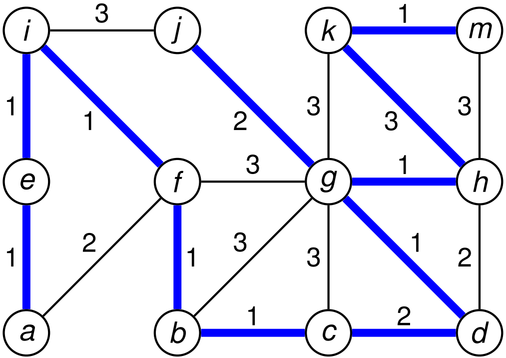

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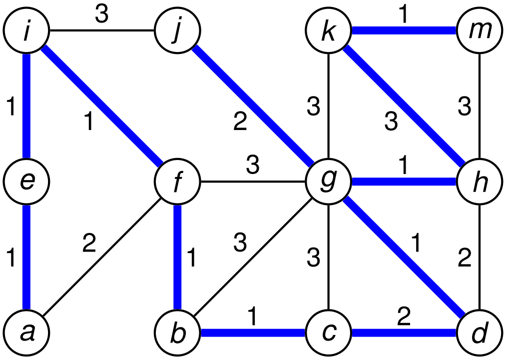

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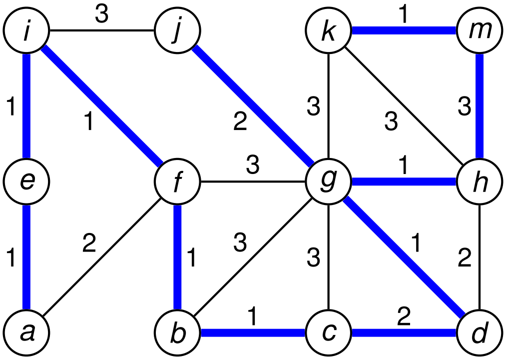

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- This is a *“greedy” algorithm*
- Does it work?

Refined Idea

Generic-MST(G, w)

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A = ∅
while A is not a spanning tree
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Refined Idea

Generic-MST(G, w)

$A = \emptyset$
while A is not a spanning tree
    find a *safe* edge $e = (u, v)$
    $A = A \cup \{e\}$

- ***Invariant:*** A is a subset of a minimum spanning tree
- A ***safe edge*** is an edge that maintains the invariant
  - e is such that, if A is a subset of a minimum spanning tree, then $A \cup \{e\}$ is also a subset of a minimum spanning tree

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  - more or less the *definition* of a greedy algorithm

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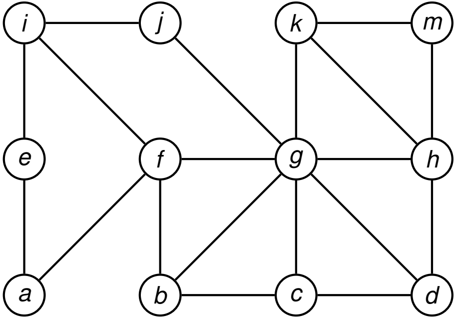

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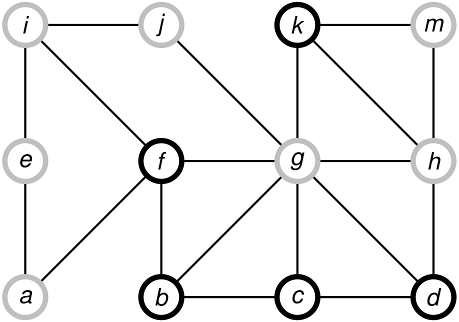

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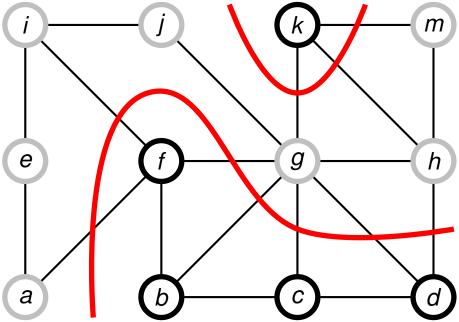

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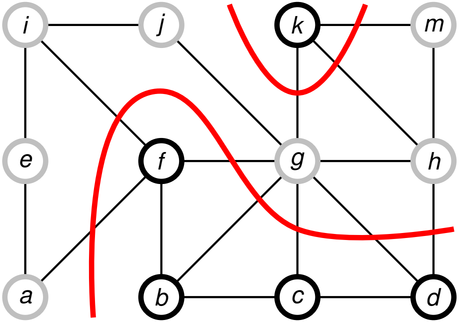

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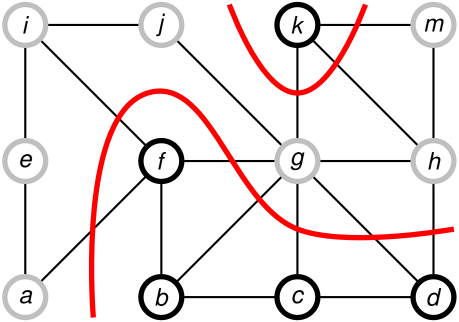

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- A cut $(S, V - S)$ *respects* a set of edges A if no edge in A crosses the cut

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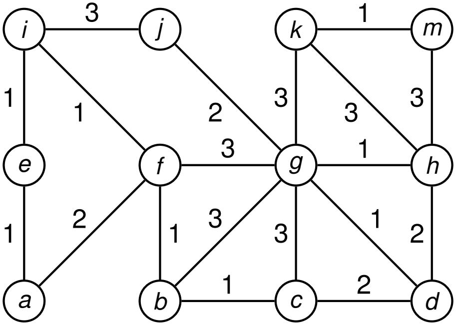

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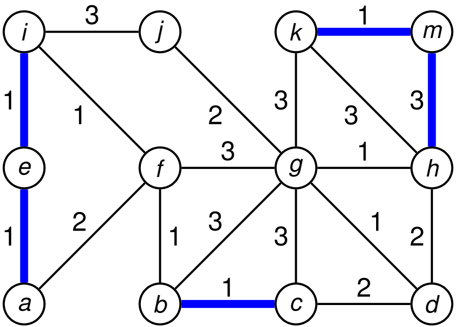

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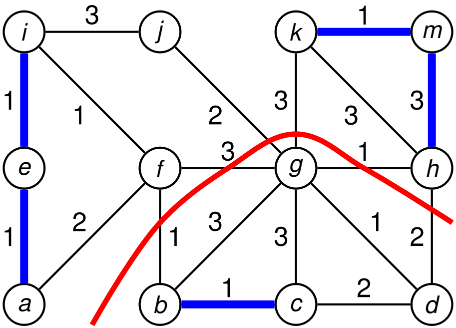

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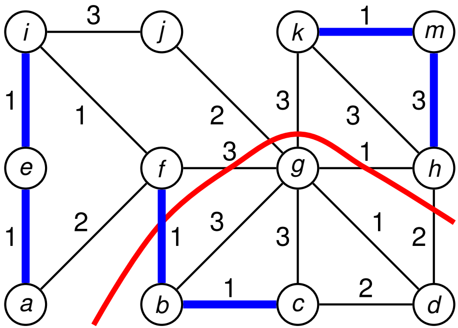

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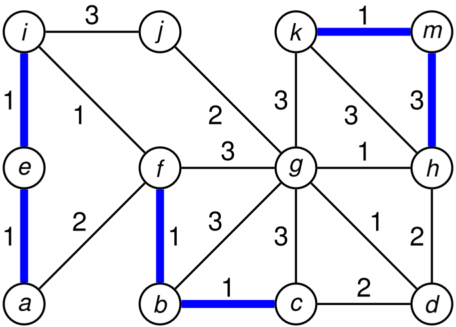

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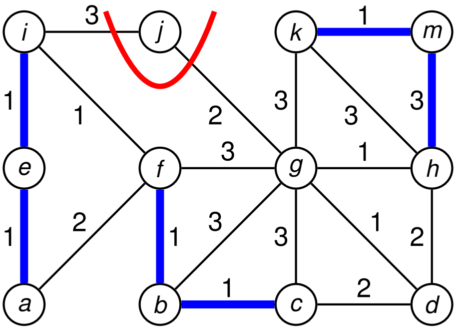

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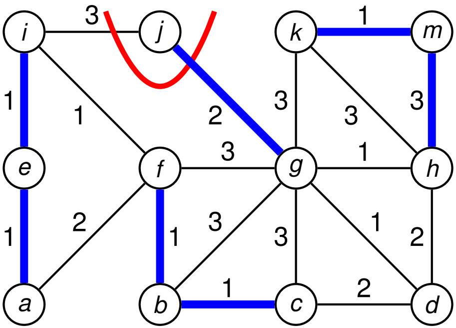

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Concrete Algorithms

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  - incrementally builds a ***forest*** A

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- Prim's algorithm (1957)
  - based on the generic minimum-spanning-tree algorithm
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- *Union*(x, y) joins the sets containing x and y

Kruskal's Algorithm

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MST-Kruskal(G, w)
A = ∅
for each vertex v ∈ V(G)
    Make-Set(v)                               // disjoint-set data structure
sort E in non-decreasing order by weight w
for each edge (u, v) ∈ E, taken in non-decreasing order by w
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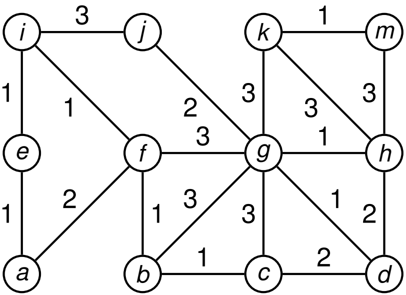

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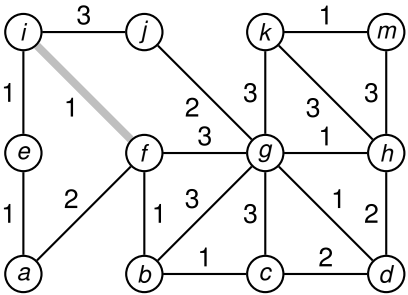

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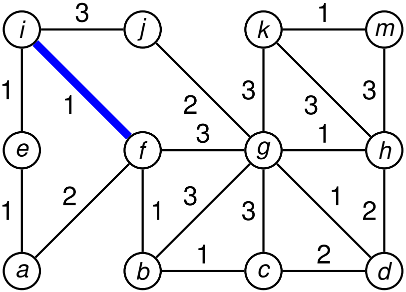

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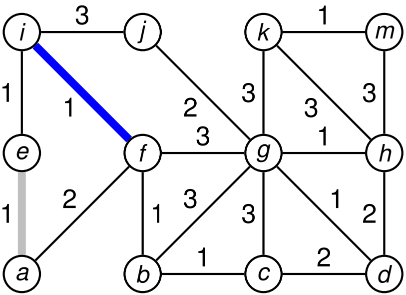

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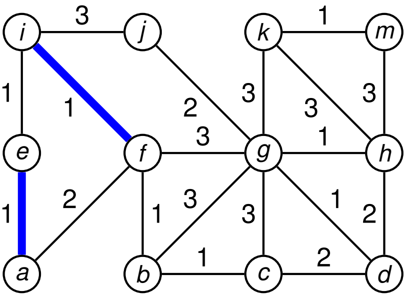

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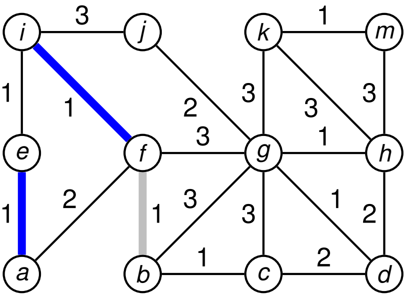

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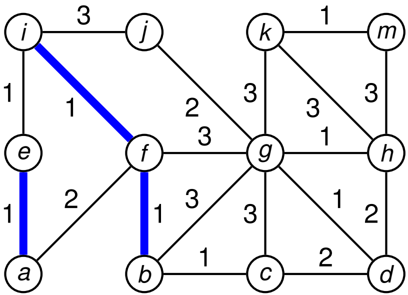

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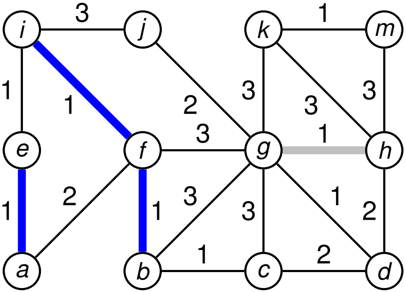

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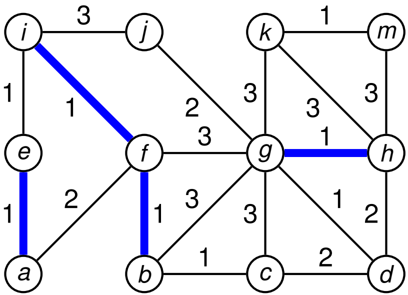

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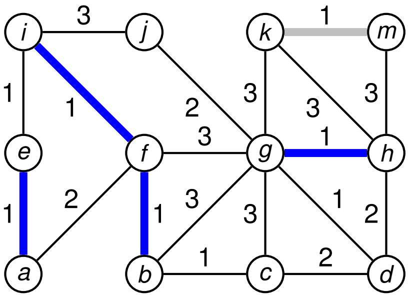

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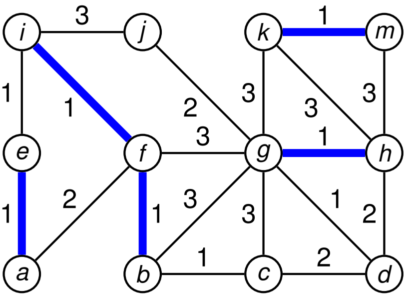

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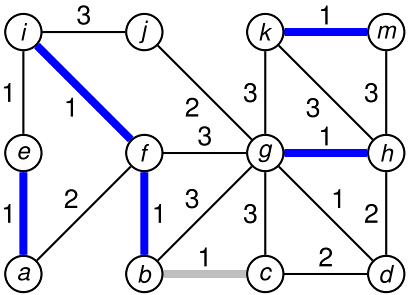

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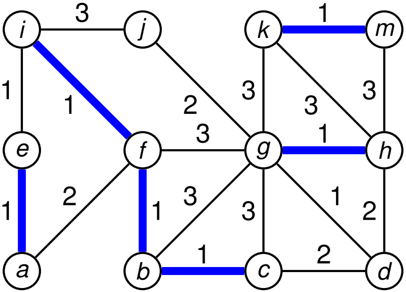

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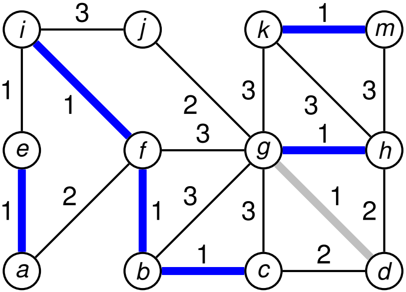

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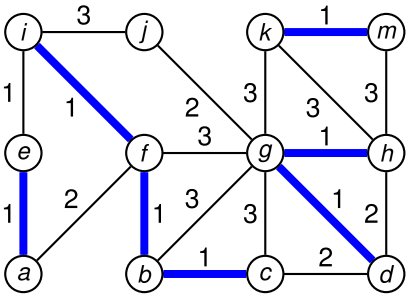

Kruskal's Algorithm

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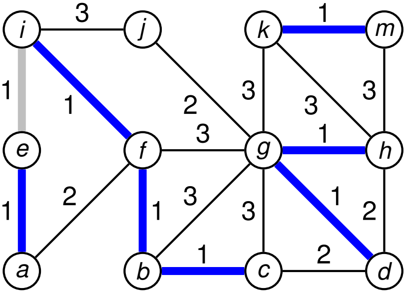

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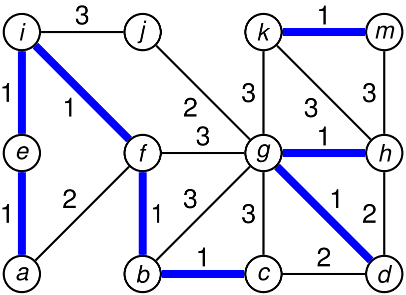

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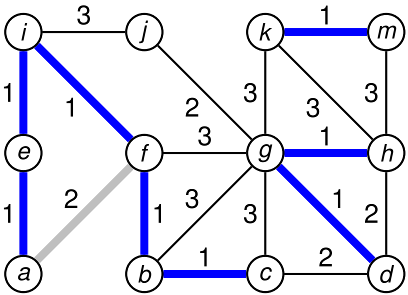

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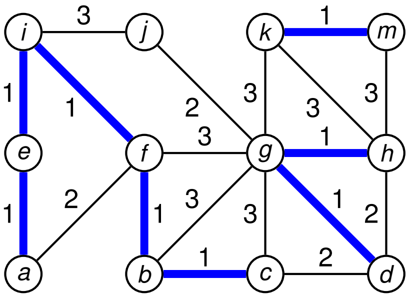

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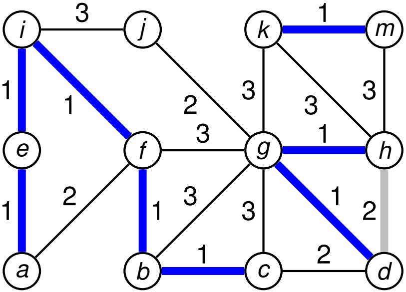

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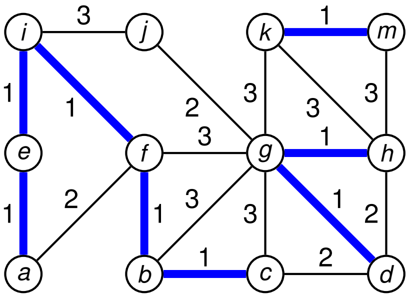

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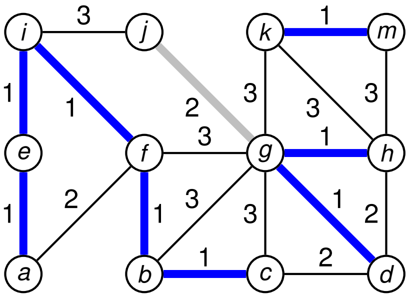

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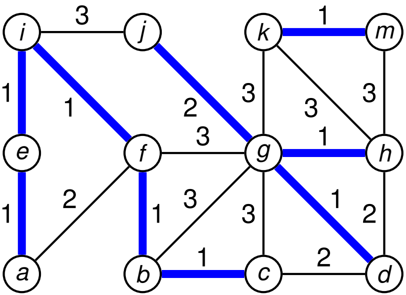

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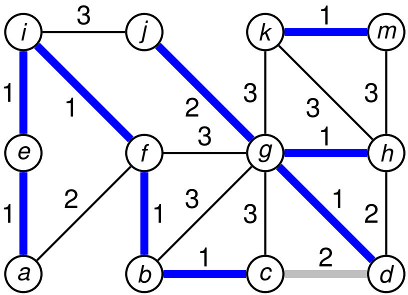

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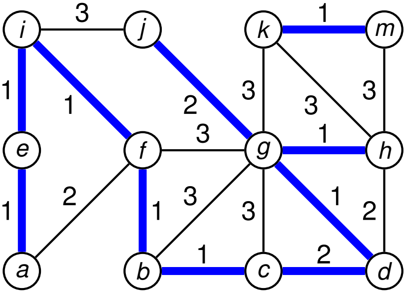

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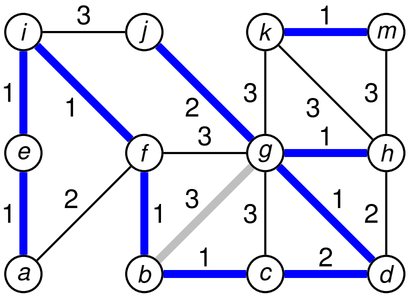

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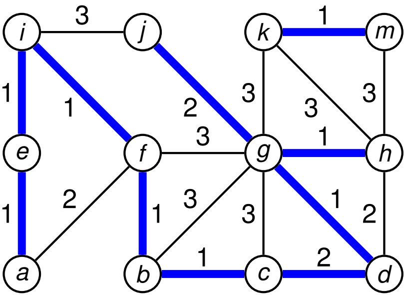

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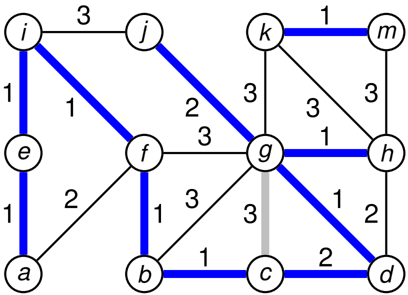

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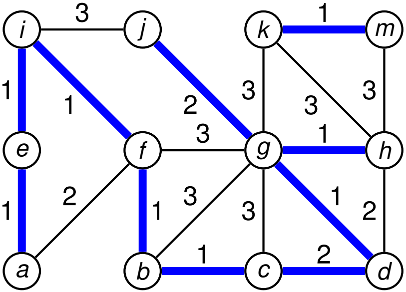

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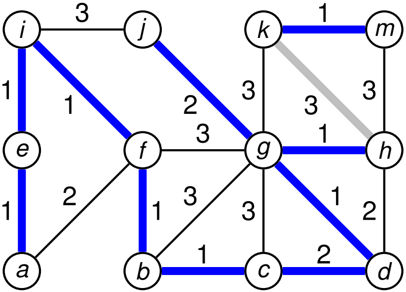

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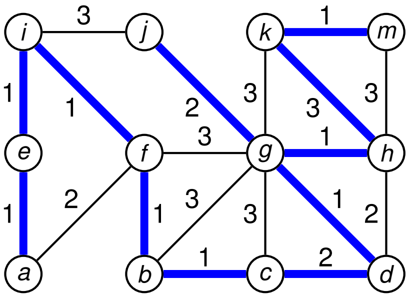

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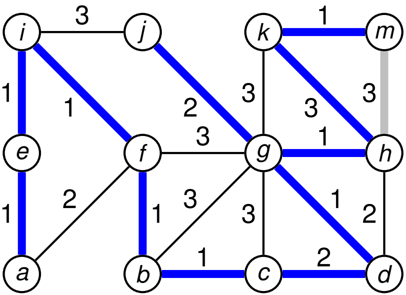

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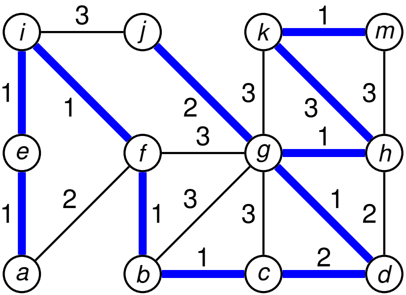

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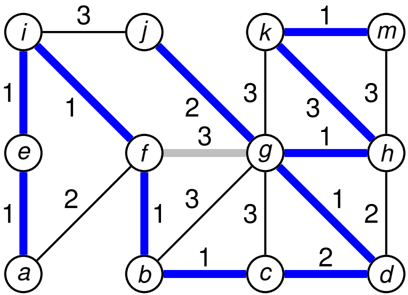

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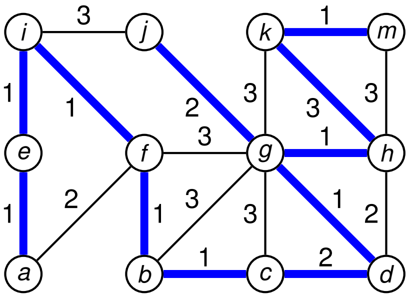

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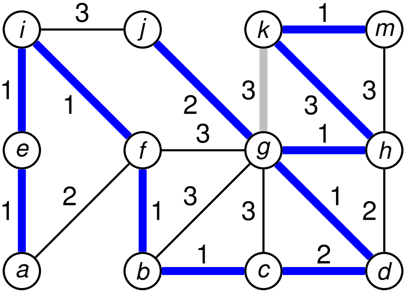

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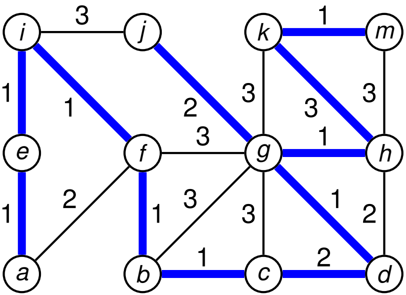

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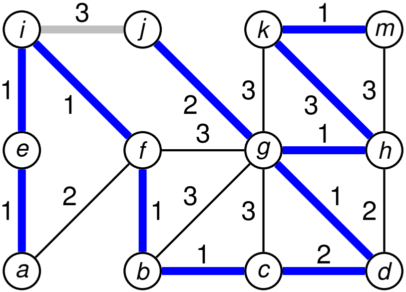

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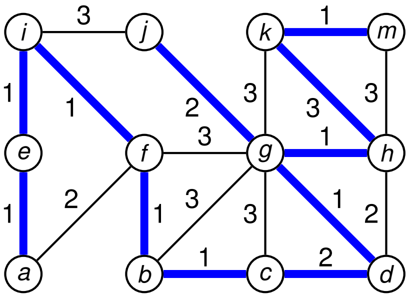

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        Union(u, v)
```

Complexity of MST-Kruskal

MST-Kruskal(G, w)

1 $A = \emptyset$
2 **for** each vertex $v \in V(G)$
3   **Make-Set**(v) // *disjoint-set* data structure
4 sort E in non-decreasing order by weight w
5 **for** each edge $(u, v) \in E$, taken in non-decreasing order by w
6   **if Find-Set**$(u) \neq$ **Find-Set**(v)
7         $A = A \cup \{(u, v)\}$
8   **Union**(u, v)

- $|V|$ times **Make-Set** (loop of line 2–3)

Complexity of MST-Kruskal

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MST-Kruskal(G, w)
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- $O(|E| \log |E|)$ for sorting E (line 4)

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- $O(|E| \log |E|)$ for sorting E (line 4)
- $2|E|$ times **Find-Set**
- $O(|E|)$ times **Union**

Prim's Algorithm

```
MST-Prim(G, w, r)
for each vertex u ∈ V(G)
    key[u] = ∞
    π(u) = nil
key[r] = 0
Q = V(G)
while Q ≠ ∅
    u = Extract-Min(Q) // min by key[u]
    for each v ∈ Adj[u]
        if v ∈ Q ∧ w(u, v) < key[v]
            π(v) = u
            key[v] = w(u, v)
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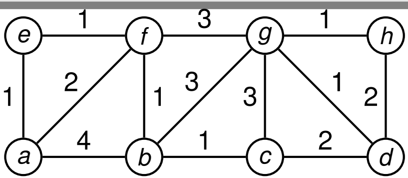

$$Q = \{(a, 0, \cdot), (b, \infty, \cdot), (c, \infty, \cdot), (d, \infty, \cdot), (e, \infty, \cdot), (f, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

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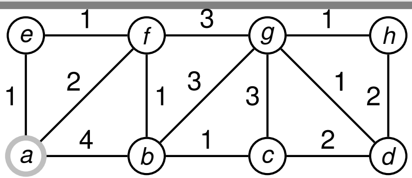

$$Q = \{(b, \infty, \cdot), (c, \infty, \cdot), (d, \infty, \cdot), (e, \infty, \cdot), (f, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

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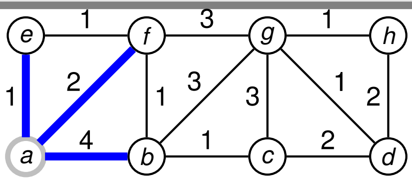

$$Q = \{(e, 1, a), (f, 2, a), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

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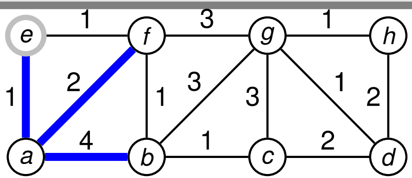
```



$$Q = \{(f, 2, a), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

```
MST-Prim(G, w, r)
for each vertex u ∈ V(G)
        key[u] = ∞
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key[r] = 0
Q = V(G)
while Q ≠ ∅
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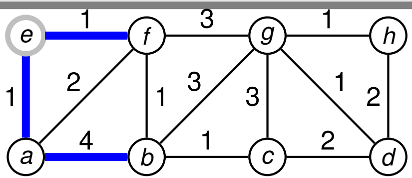
        for each v ∈ Adj[u]
                if v ∈ Q ∧ w(u, v) < key[v]
                        π(v) = u
                        key[v] = w(u, v)
```



$$Q = \{(f, 1, e), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

Prim's Algorithm

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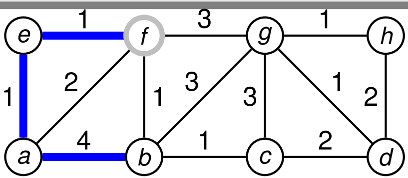
```



$Q = \{(b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

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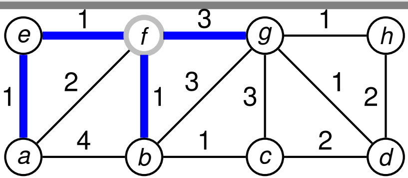

$$Q = \{(b, 1, f), (g, 3, f), (c, \infty, \cdot), (d, \infty, \cdot), (h, \infty, \cdot)\}$$

Prim's Algorithm

```
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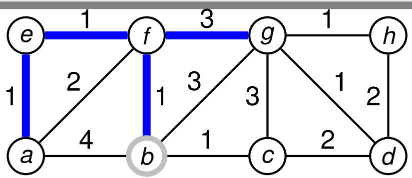

$Q = \{(g, 3, f), (c, \infty, \cdot), (d, \infty, \cdot), (h, \infty, \cdot)\}$

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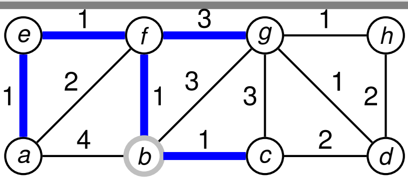

$Q = \{(c, 1, b), (g, 3, f), (d, \infty, \cdot), (h, \infty, \cdot)\}$

Prim's Algorithm

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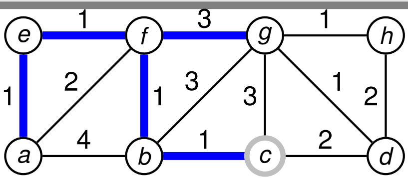

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Prim's Algorithm

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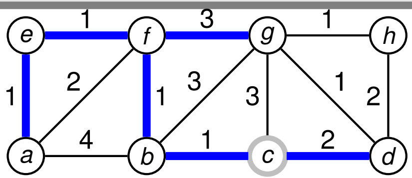

$Q = \{(d, 2, c), (g, 3, f), (h, \infty, \cdot)\}$

```
MST-Prim(G, w, r)
for each vertex u ∈ V(G)
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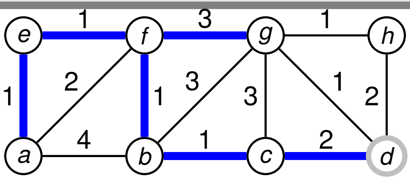
```



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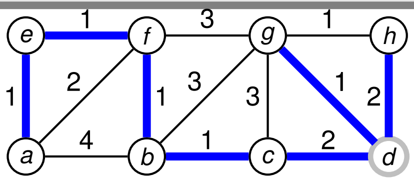

$$Q = \{(g, 1, d), (h, 2, d)\}$$

MST-Prim(G, w, r)

```
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    key[u] = ∞
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Q = V(G)
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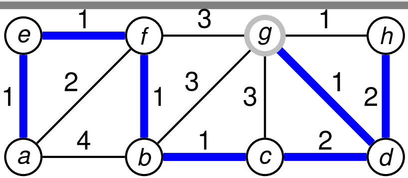

$Q = \{(h, 2, d)\}$

Prim's Algorithm

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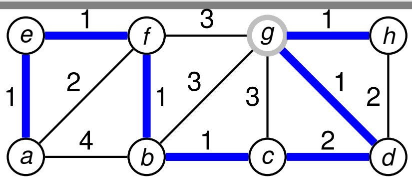

$Q = \{(h, 1, g)\}$

Prim's Algorithm

MST-Prim(G, w, r)

```
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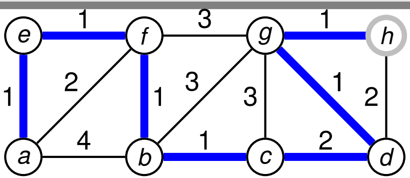

$Q = \{\}$

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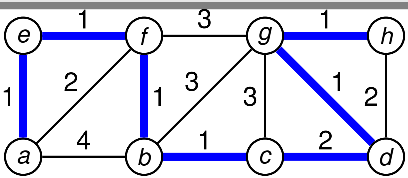

$Q = \{\}$