Dynamic Programming

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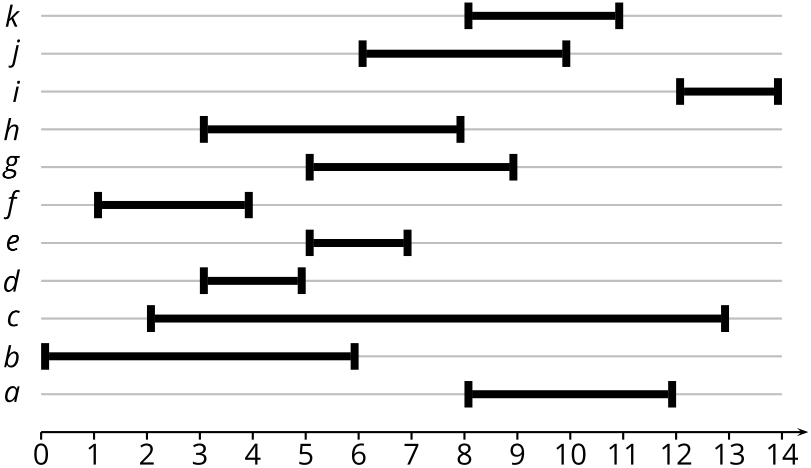
May 21, 2019

Outline

- Examples
- Dynamic programming strategy
- More examples

Activity-Selection Problem



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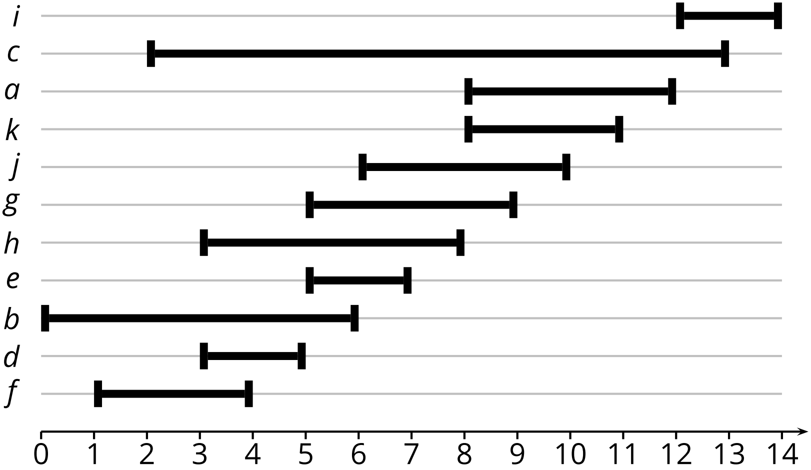

Greedy choice: ***earliest finish time***

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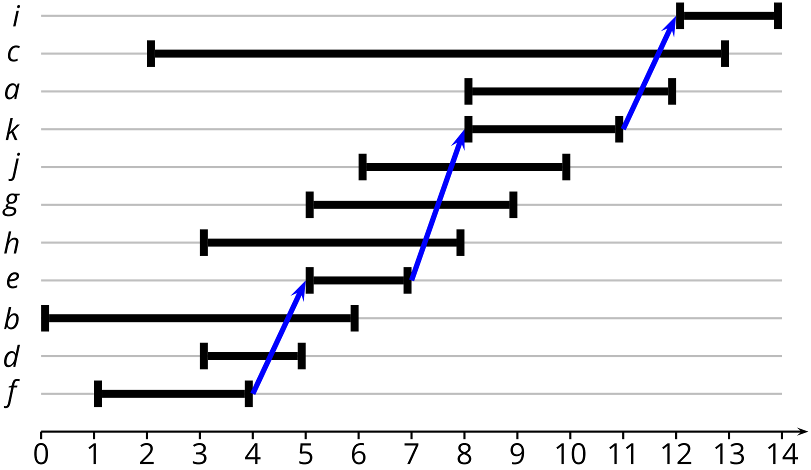

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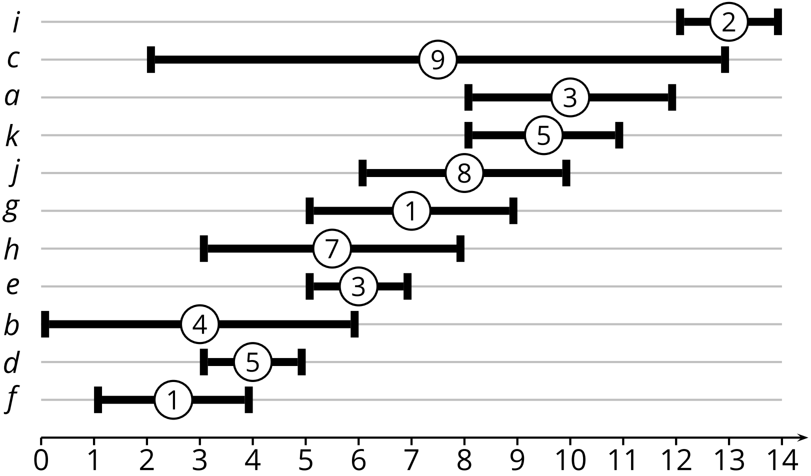

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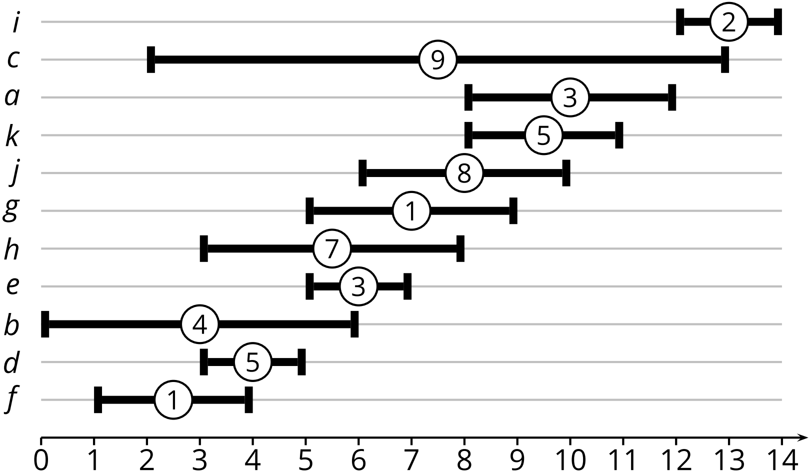

Is the earliest-finish greedy choice still optimal?

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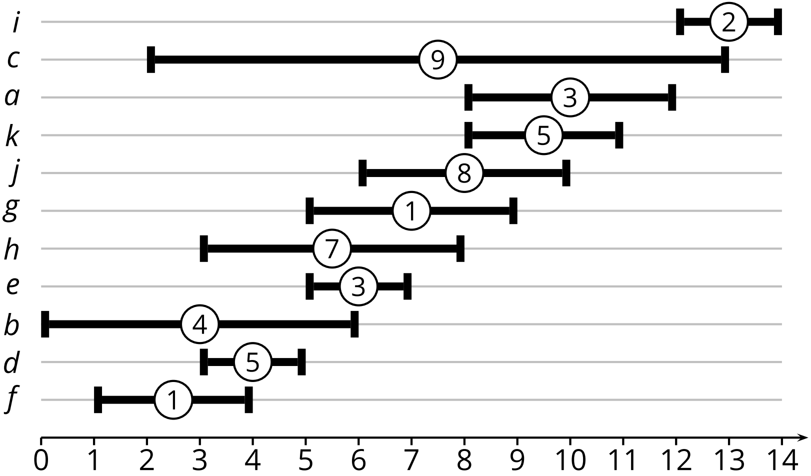
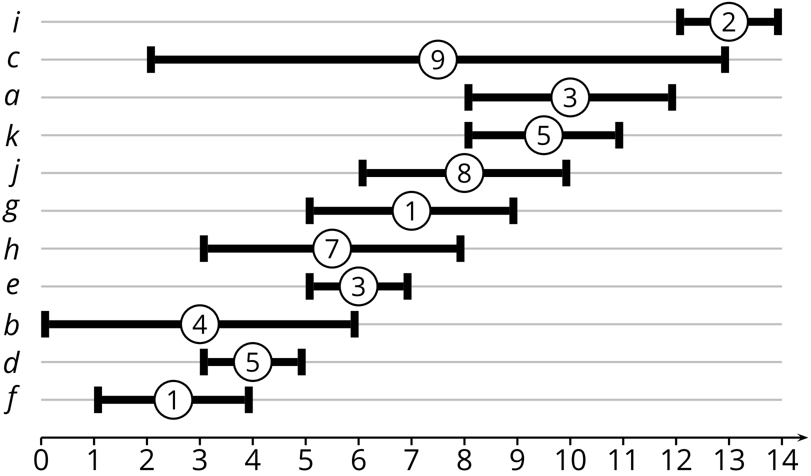
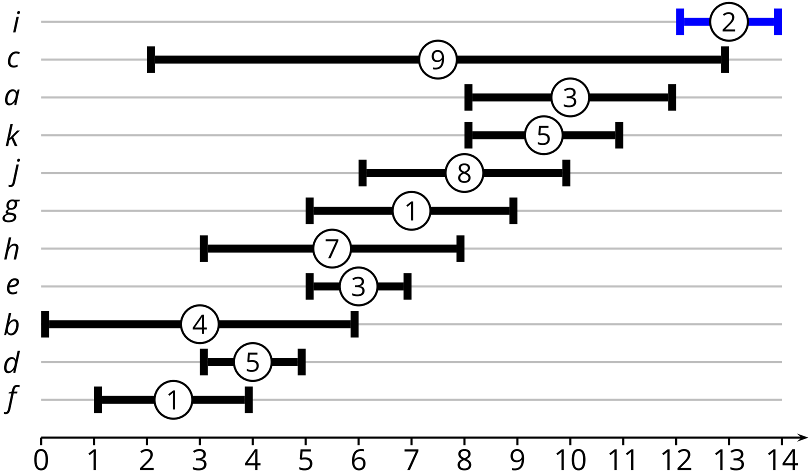


Is the earliest-finish greedy choice still optimal?

Is *any* greedy choice optimal?

Case 1





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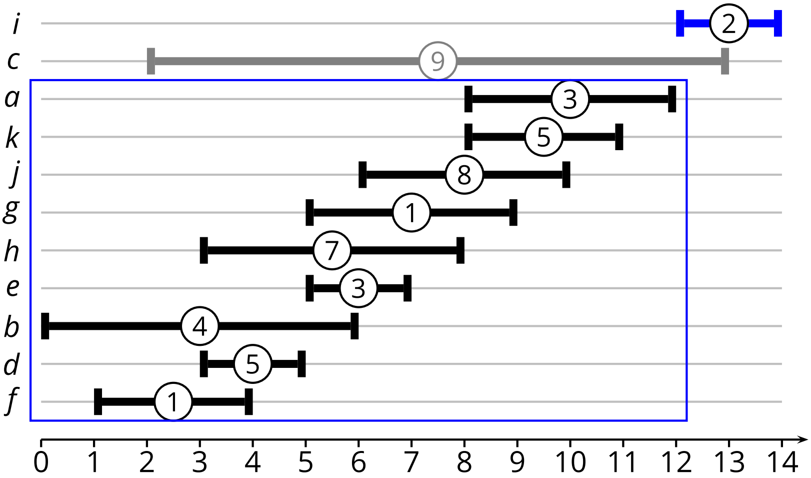

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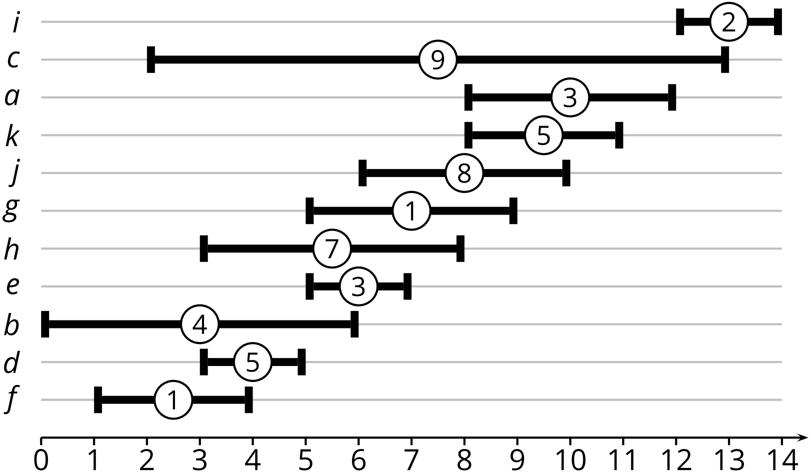

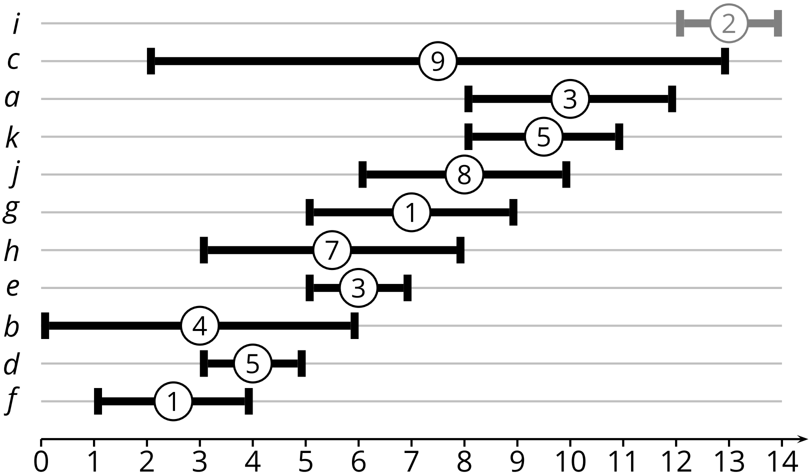

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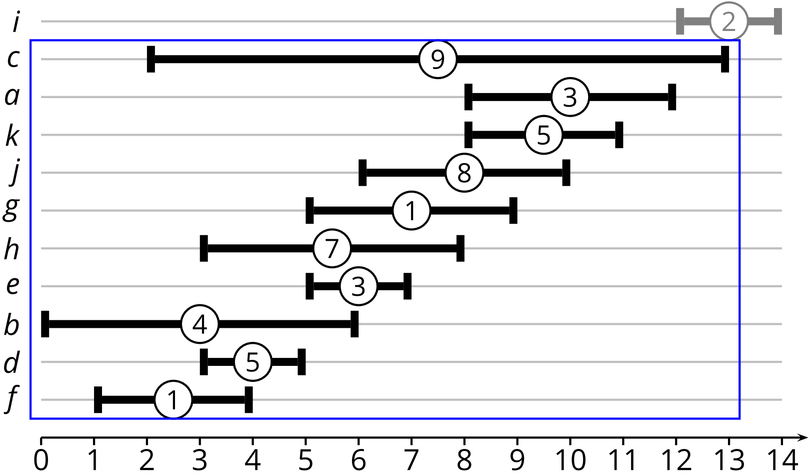

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Bellman-Ford Algorithm

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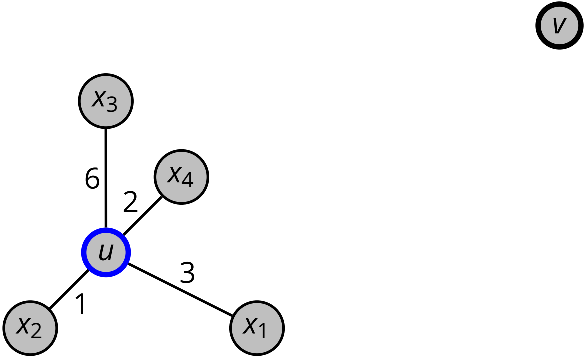

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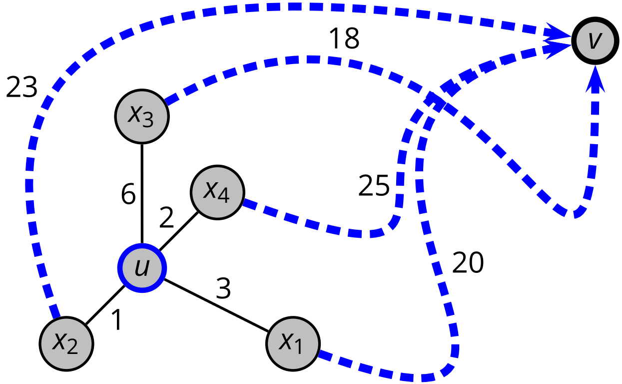

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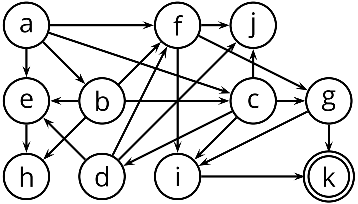

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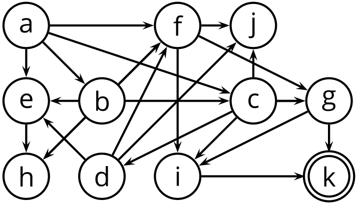

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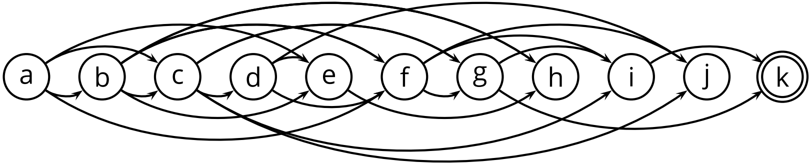

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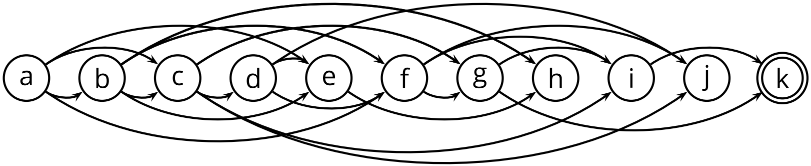

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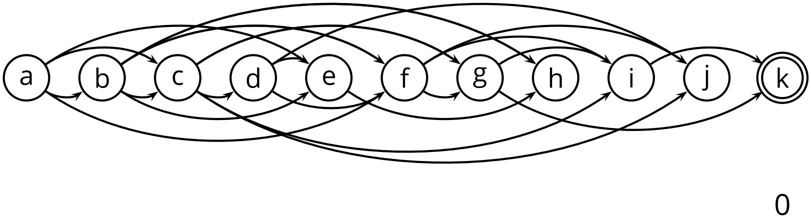

Shortest Paths on DAGs (2)

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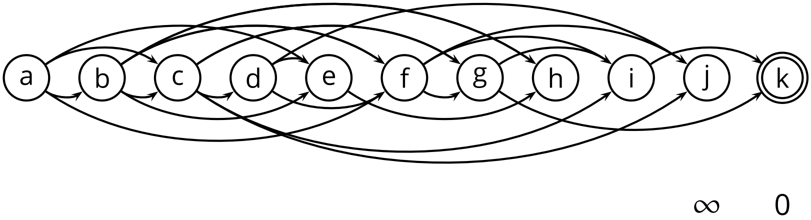

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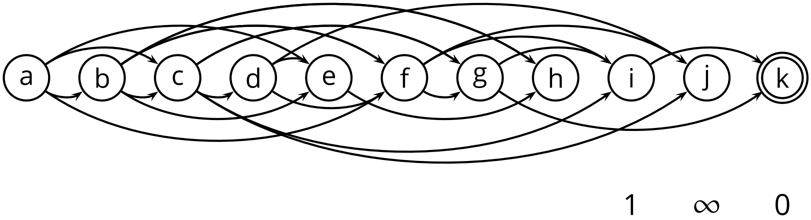

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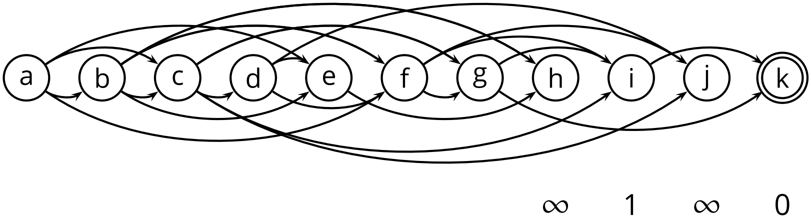

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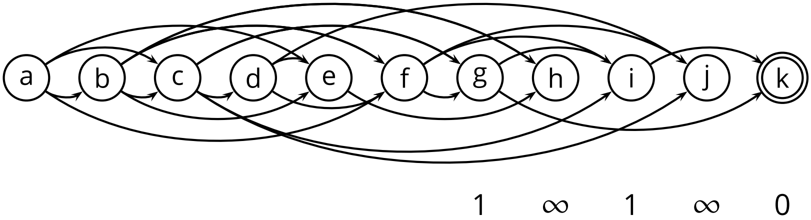

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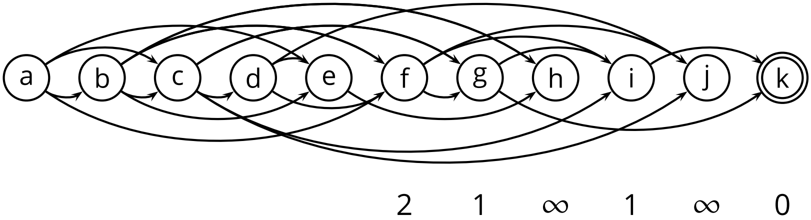

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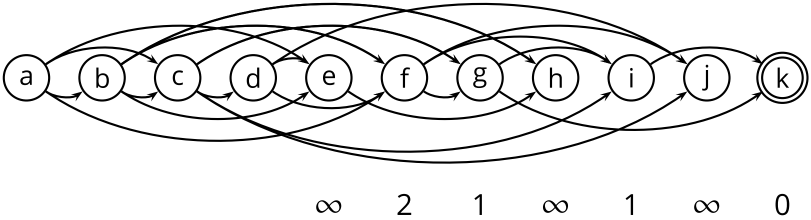

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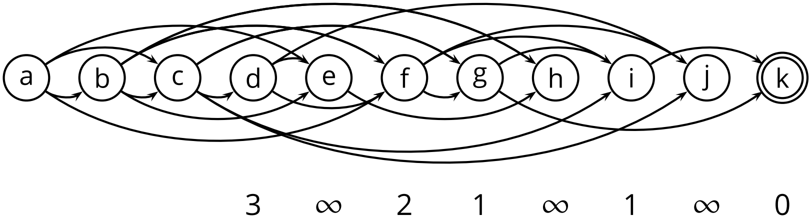

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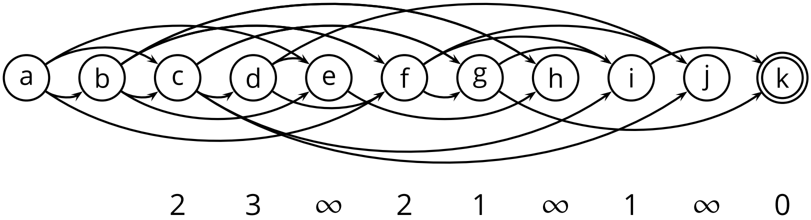

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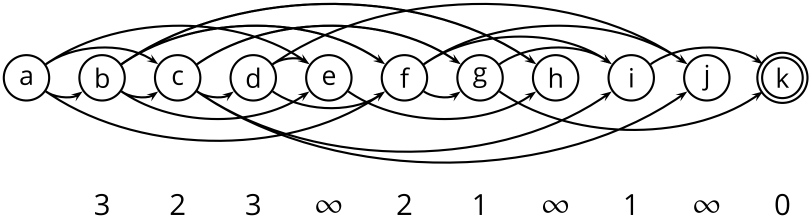

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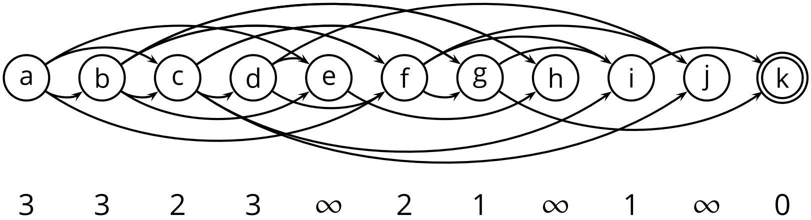

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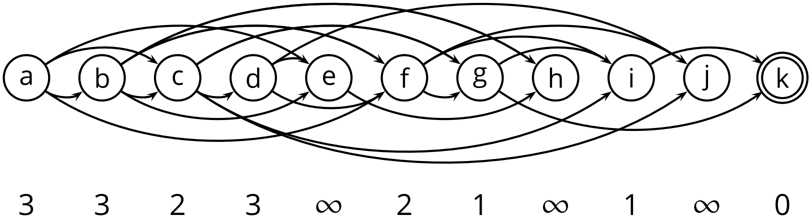

- Since G is a DAG, computing D_y with $y \in Adj(x)$ can be considered a *subproblem* of computing D_x
  - we build the solution bottom-up, storing the subproblem solutions

Longest Increasing Subsequence

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- Given a sequence of numbers $a_1, a_2, \ldots, a_n$, an *increasing subsequence* is any subset $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, and such that

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  A maximal-length subsequence is

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- Combining the subproblems

$$L(j) = 1 + \max\{L(i) \mid i < j \wedge a_i < a_j\}$$

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  - derive the solution from (one of) the solutions to the subproblems

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  - **exercise:** find a counter-example

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- Divide-and-conquer splits the problem into ***independent subproblems***
  - in dynamic programming, subproblems typically overlap
  - pretty much the same argument as above

Dynamic Programming vs. Greedy

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- Dynamic programming: ***more general***
  - does not need the greedy-choice property
  - typically looks at several subproblems
    - "dynamically" choose one of them to obtain a global solution
  - typically works bottom-up
  - typically reuses solutions of the subproblems

Typical Subproblem Structures

- Prefix/suffix subproblems
  - *Input:* $x_1, x_2, \ldots, x_n$
  - *Subproblem:* $x_1, x_2, \ldots, x_i$, with $i < n$
  - $O(n)$ subproblems

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C	a	r	z	a	n			i	g	a
J	a		z	a	y	e	r	i		

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$\Downarrow$		$-$			$\Downarrow$	$+$	$+$		$-$	$-$
J					y	e	r			
C	a	r	z	a	n			i	g	a
J	a		z	a	y	e	r	i		

Edit Distance (2)

- Align the two strings x and y, possibly inserting “gaps” between letters
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- This suggests a way to combine the subproblems; let $\mathit{diff}(i, j) = 1$ iff $x[i] \neq y[j]$ or 0 otherwise

$$
\begin{aligned}
E(i,j) = \min\{ & 1 + E(i-1, j), \\
& 1 + E(i, j-1), \\
& \mathit{diff}(i,j) + E(i-1, j-1)\}
\end{aligned}
$$

- Problem definition
  - *Input:* a set of n objects with their weights $w_1, w_2, \ldots w_n$ and their values $v_1, v_2, \ldots v_n$, and a maximum weight W
  - *Output:* a subset K of the objects such that $\sum_{i \in K} w_i \leq W$ and such that $\sum_{i \in K} v_i$ is maximal

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Dynamic-programming solution

- let $K(w, j)$ be the maximum value achievable at maximum capacity w using the first j items (i.e., items $1 \ldots j$)
- considering the jth element, we can either "use it or loose it," so

$$K(w, j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}$$

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    return 0
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- ***Recursion solves the same problem over and over again***

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- **Idea:** “cache” the results

```
FIBONACCI(n)
if n == 0
        return 0
elseif n == 1
        return 1
elseif (n, x) ∈ H // a hash table H “caches” results
        return x
else x = FIBONACCI(n − 1) + FIBONACCI(n − 2)
        INSERT(H, n, x)
        return x
```

- Idea also known as *memoization*

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  1. start with the greedy choice
  2. add the solution to the remaining subproblem

  A nice tail-recursive algorithm

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  3. in practice, solve the subproblems bottom-up

Exercise

- **Puzzle 0:** is it possible to insert some ‘+’ signs in the string “213478” so that the resulting expression would equal 214?

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  - Yes, because 2 + 134 + 78 = 214
- **Puzzle 1:** is it possible to insert some '+' signs in the strings of digits to obtain the corresponding target number?

digits	*target*
646805736141599100791159198	472004
6152732017763987430884029264512187586207273294807	560351
48796142803774467559157928	326306
195961521219109124054410617072018922584281838218	7779515