Basics of Complexity Analysis: The RAM Model and the Growth of Functions

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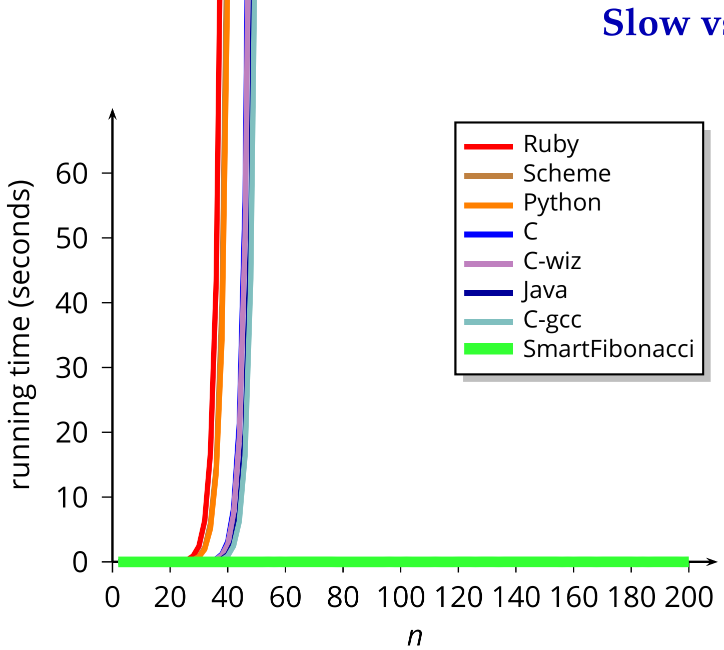
Outline

- Informal analysis of two Fibonacci algorithms
- The *random-access machine* model
- Measure of complexity
- Characterizing functions with their asymptotic behavior
- Big-*O*, omega, and theta notations

Slow vs. Fast Fibonacci



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  - in general
  - in a way that is ***specific to the algorithms***
  - but ***independent of implementation details***

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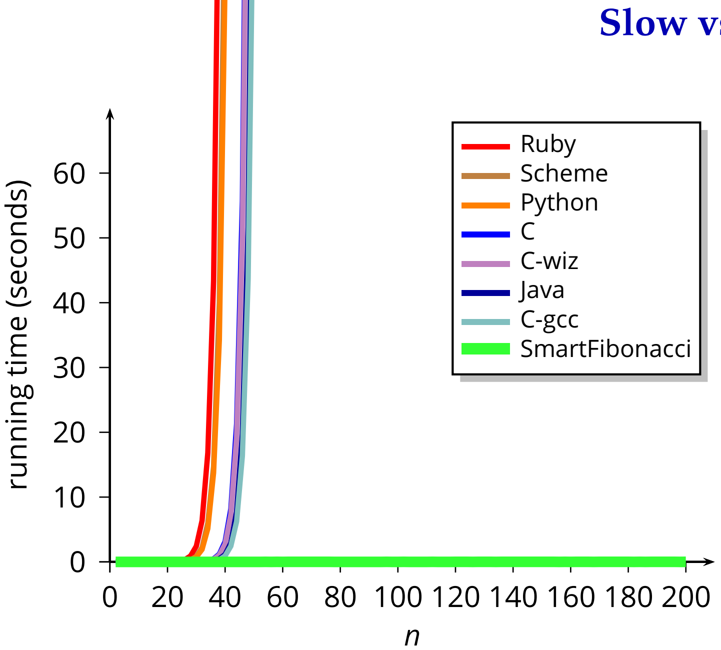

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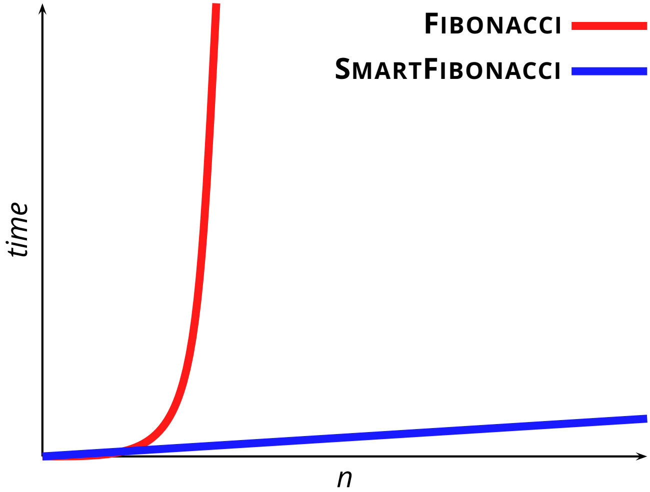

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  - load/store: assignment, use of a variable
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- A ***basic step*** in the RAM model takes a ***constant time***

Analysis in the RAM Model

```
SmartFibonacci(n)
if n == 0
        return 0
elseif n == 1
        return 1
else pprev = 0
        prev = 1
        for i = 2 to n
                f = prev + pprev
                pprev = prev
                prev = f
return f
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cost	*times* $(n > 1)$
c_1	1
c_2	0
c_3	1
c_4	0
c_5	1
c_6	1
c_7	n
c_8	$n-1$
c_9	$n-1$
c_{10}	$n-1$
c_{11}	1

$$T(n) = c_1 + c_3 + c_5 + c_6 + c_{11} + nc_7 + (n-1)(c_8 + c_9 + c_{10})$$

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$$\boxed{T(n) = nC_1 + C_2 \quad \Rightarrow T(n) \text{ is a } \textit{linear function} \text{ of } n}$$

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FIND(A, x)
for i = 1 to length(A)
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$$T(n) = Cn$$

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FindEquals(A)
for i = 1 to length(A) − 1
    for j = i + 1 to length(A)
        if A[i] == A[j]
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$$T(n) = C\frac{n(n-1)}{2}$$

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  - we also ignore the specific *value* c_i, and in fact ***we ignore every constant cost factor***

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    We write

    $$\boxed{T(n) = \Theta(n^2)}$$

    and say that "$T(n)$ is theta of n-squared"

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  - $A(n-1) = A(n^2)$ for all n

- If $f(n)$ is such that $f(n) = kA(g(n))$ for all n sufficiently large and for some constant $k > 0$, then we say that

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- If $f(n) = O(g(n))$ then we can also say that $g(n)$ asymptotically *dominates* $f(n)$, which we can also write as

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- When $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ we also write

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  - $\pi(n) = \Omega(1)$ trivial ***lower bound***
  - $\pi(n) = \Theta(n/\log n)$ non-trivial ***tight bound***

- The idea of the O, Ω, and Θ notations is very often to characterize a function that is *not completely known*

  Example:

  Let $\pi(n)$ be the number of *primes* less than or equal to n

  What is the asymptotic behavior of $\pi(n)$?

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  In fact, the fundamental *prime number theorem* says that

$$\lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = 1$$

Θ-Notation

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- Given a function $g(n)$, we define the *family of functions* $\Theta(g(n))$

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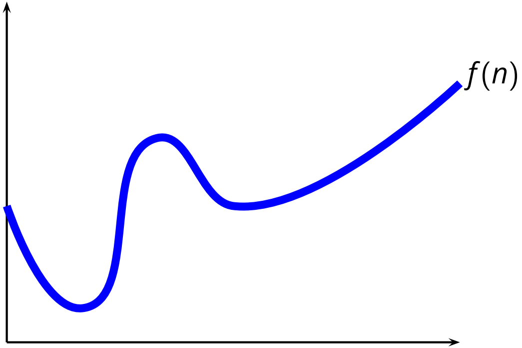

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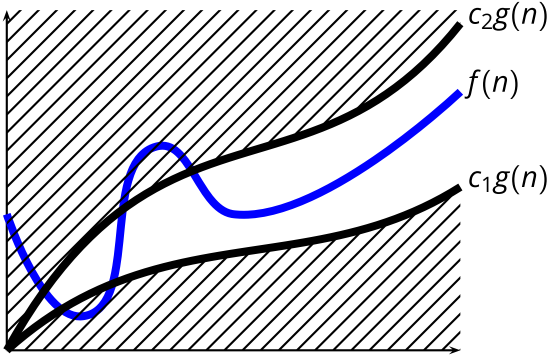

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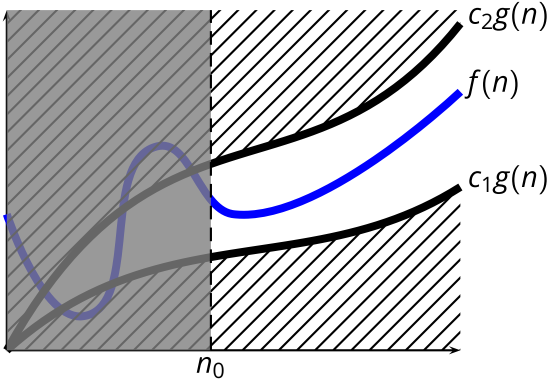

$$\Theta(g(n)) = \{f(n) : \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0 \\ : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0\}$$

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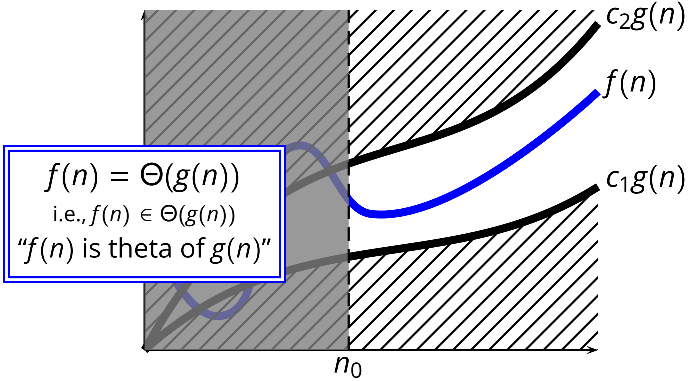

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Examples

- $T(n) = n^2 + 10n + 100$

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- $T(n) =$ complexity of **SmartFibonacci** $\quad \Rightarrow T(n) = \Theta(n)$
- We characterize the behavior of $T(n)$ *in the limit*
- The Θ-notation is an ***asymptotic notation***

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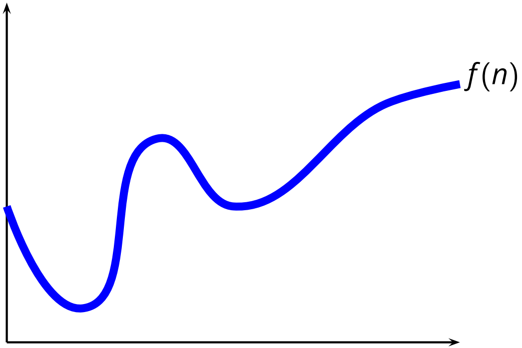

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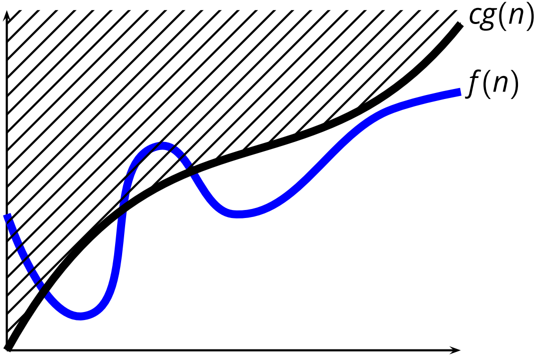

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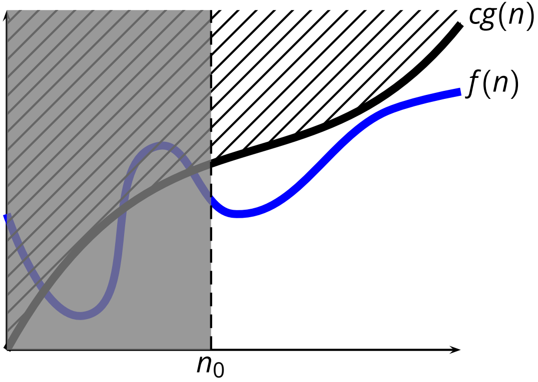

$$O(g(n)) = \{f(n) : \exists c > 0, \exists n_0 > 0 \\ : 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

O-Notation

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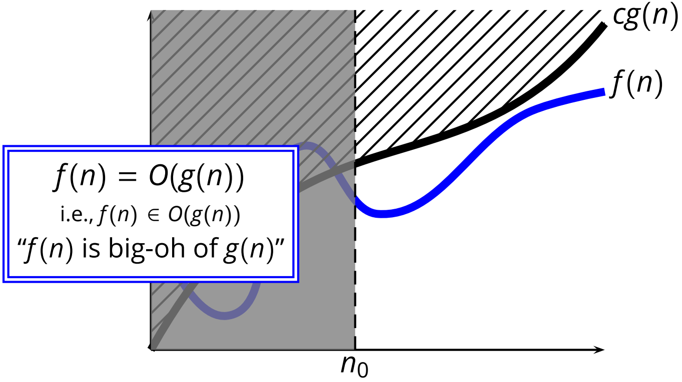

$$\begin{aligned} O(g(n)) = \{f(n) &: \exists c > 0, \exists n_0 > 0 \\ &: 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\} \end{aligned}$$

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Examples

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Example

- So, what is the complexity of **FINDEQUALS**?

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FINDEQUALS(A)
for i = 1 to length(A) − 1
    for j = i + 1 to length(A)
        if A[i] == A[j]
            return TRUE
return FALSE
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$$T(n) = \Theta(n^2)$$

- $n = \mathit{length}(A)$ is the ***size of the input***
- we measure the ***worst-case complexity***

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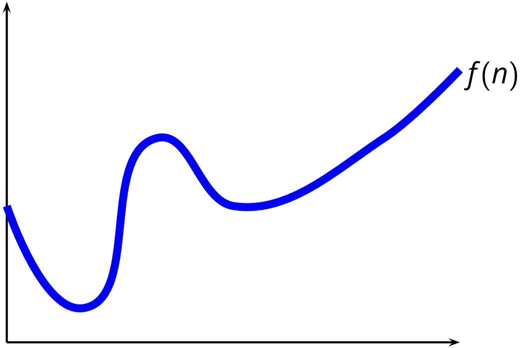

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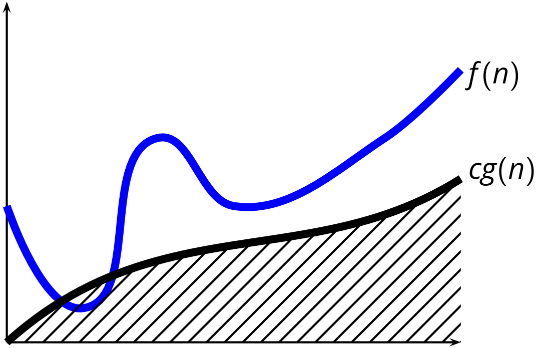

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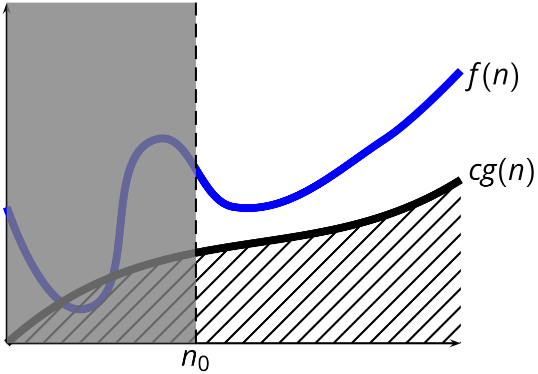

$$\begin{aligned}\Omega(g(n)) = \{f(n) &: \exists c > 0, \exists n_0 > 0 \\ &: 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\}\end{aligned}$$

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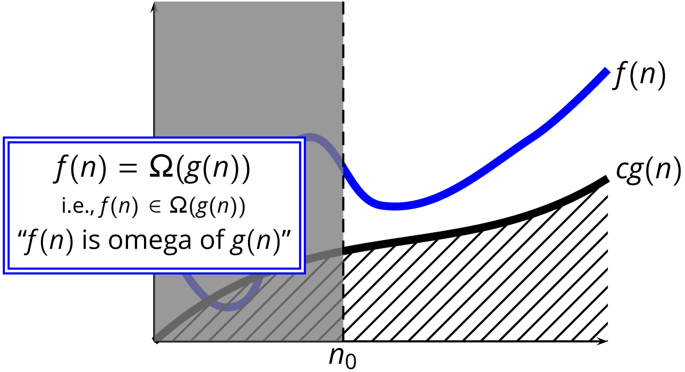

$$f(n) = \Omega(g(n))$$

i.e., $f(n) \in \Omega(g(n))$

"$f(n)$ is omega of $g(n)$"

$$\Omega(g(n)) = \{f(n) : \exists c > 0, \exists n_0 > 0$$
$$: 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0\}$$

- *Theorem:* for any two functions $f(n)$ and $g(n)$,
  $f(n) = \Omega(g(n)) \wedge f(n) = O(g(n)) \Leftrightarrow f(n) = \Theta(g(n))$

Θ, O, and Ω as Relations

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- The above theorem can be interpreted as saying

$$f \geq g \wedge f \leq g \Leftrightarrow f = g$$

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