Basic Elements of Complexity Theory

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Outline

- Basic complexity classes
- Polynomial reductions
- NP-completeness

Polynomial Time

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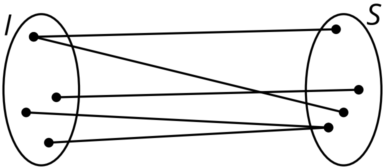
Abstract Problems

- An ***abstract problem*** Q is a binary relation between a set I of problem ***instances*** and a set S of ***solutions***



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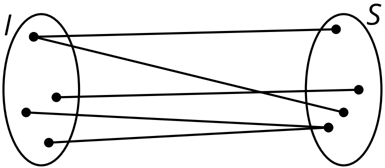

- A ***concrete problem*** Q is one where I and S are the set of binary strings $\{0, 1\}^*$
  - for all practical purposes, instances and solutions can be ***encoded*** as binary strings (i.e., mapped into $\{0, 1\}^*$)
  - we consider only sensible encodings…

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Primality Testing

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Example: shortest path in a graph

$$G = (V = \{a, b, c, \ldots\}, E = \{(a, c), \ldots\}), a, z \longrightarrow a, c, \ldots, z$$

  - *input:* a graph G, a start vertex (a), and an end vertex (z)
  - *output:* a sequence of vertexes $a, c, \ldots, z$

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Shortest path as a decision problem

$$G = (V = \{a, b, c, \ldots\}, E = \{(a, c), \ldots\}), a, z, 10 \longrightarrow 1$$

- *input:* a graph G, a start vertex (a), an end vertex (z), and a path length (10)
- *output:* 1 if there is a path of (at most) the given length

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- An optimization problem is ***not much harder*** than the corresponding decision problem
  - having a solution to the decision problem does not give an immediate solution to the optimization problem
  - but we can typically use the decision problem as a subroutine in some kind of (binary) search to solve the corresponding optimization problem

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  - parsing a Java program
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- **Example:** ***Vertex cover*** (decision variant)
  - *Input:* A graph $G = (V, E)$ and a number K
  - *Output:* 1, if there is set S of at most k vertices such that for every edge $e = (u, v) \in E$, $u \in S$ or $v \in S$ (or both); 0 otherwise

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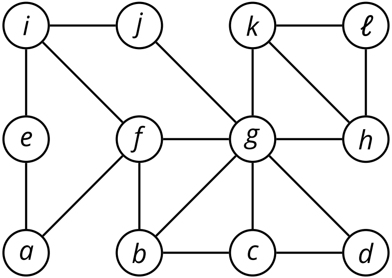

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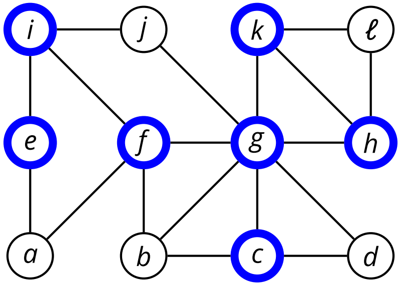

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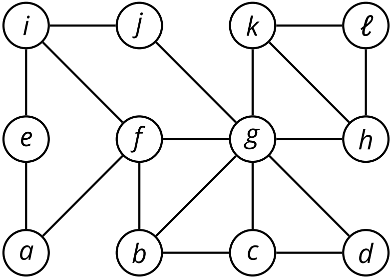

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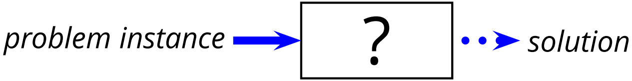

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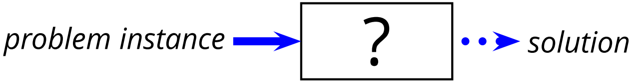

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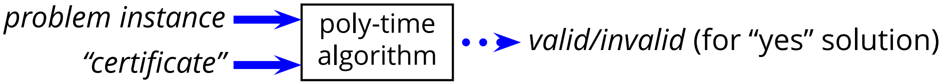

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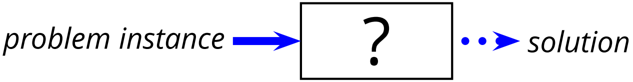

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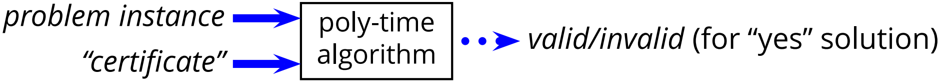

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The Complexity Class NP

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- *polynomial-time solvable* $\Longrightarrow$ *polynomial-time verifiable*

$$\mathrm{P} \subseteq \mathrm{NP}$$

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$$\mathrm{P} = \mathrm{NP}?$$

- Most theoretical computing scientists *believe* that $\mathrm{P} \neq \mathrm{NP}$
- ***Finding a solution to a problem is believed to be inherently more difficult than verifying a given solution or a proof of a solution***

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  - we don't know

Reduction

Reduction

- In our theory of complexity we want to show that a problem is *just as hard as another problem*

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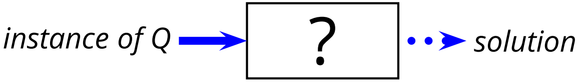
- We do that with ***polynomial-time reductions***



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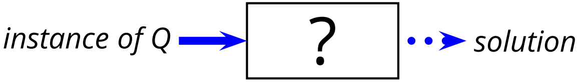

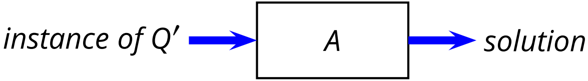

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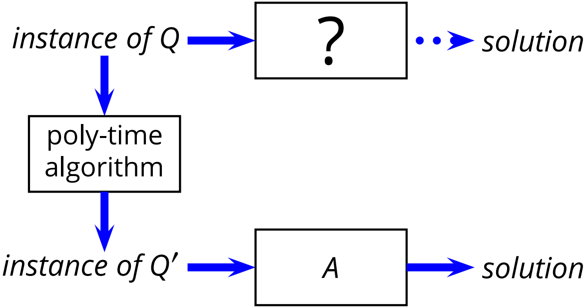

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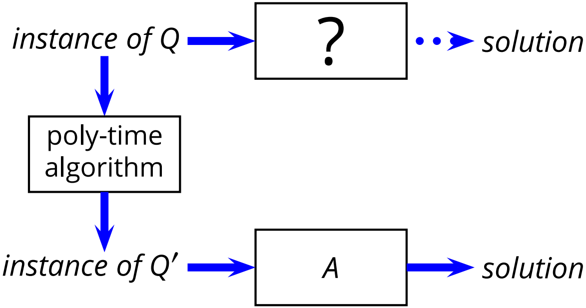

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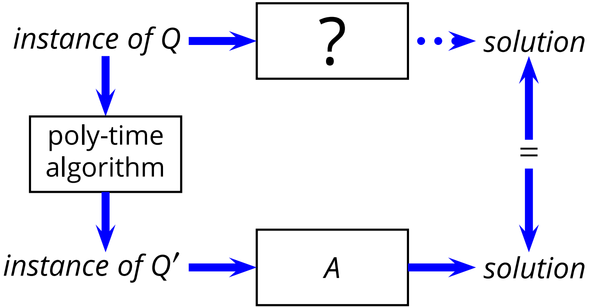

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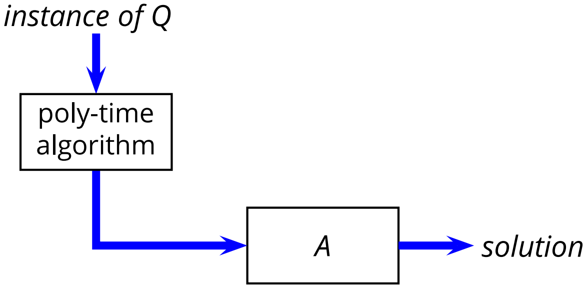
Reduction (2)

- Solution by polynomial-time reductions to a solvable problem



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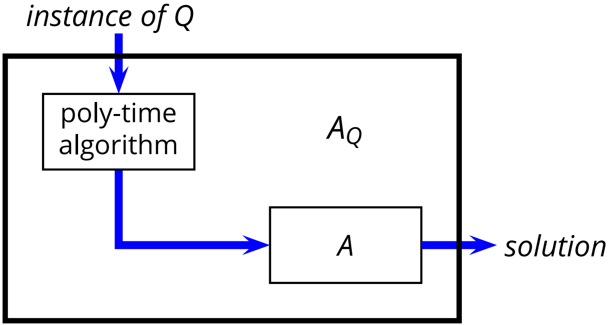

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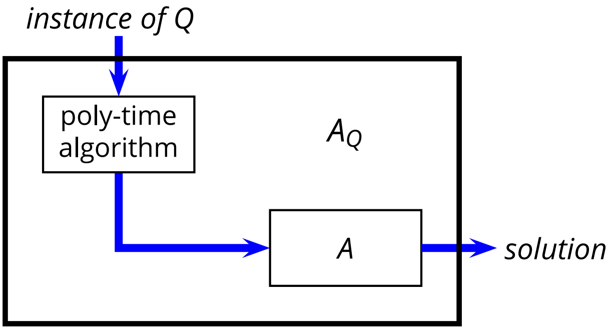

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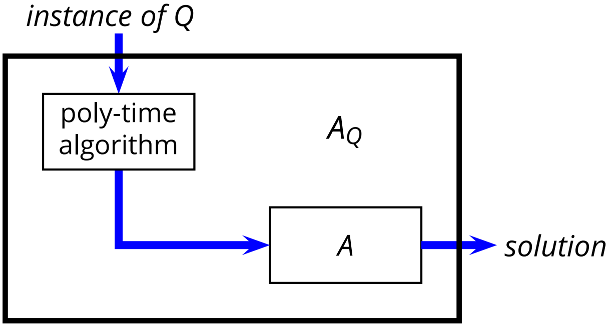

  - if A is polynomial-time, then of A_Q is also polynomial time
  - therefore if $Q' \in P$, then $Q \in P$

Example: 2-CNF-SAT

2-CNF-SAT problem

Input:

- f is a Boolean formula of n (Boolean) variables $x_1, x_2, \ldots, x_n$
- f is in *conjunctive normal form (CNF)*, so $f = C_1 \wedge C_2 \wedge \cdots \wedge C_k$
- every *clause* C_i of f contains exactly *two* literals (a variable or its negation)

Output: 1 iff f is satisfiable

- there is an assignment of variables that satisfies f

Example:

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2)$$

2-CNF-SAT to Implicative Form

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- Consider each clause C_i

$$(a \vee b) \equiv (\neg a \Rightarrow b) \equiv (\neg b \Rightarrow a)$$

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- **Example:**

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3)$$

is equivalent to

$$(\neg x_1 \Rightarrow \neg x_3) \wedge (x_3 \Rightarrow x_1) \wedge (x_2 \Rightarrow x_3) \wedge (\neg x_3 \Rightarrow \neg x_2)$$

2-CNF-SAT to Graph Reachability

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2)$$

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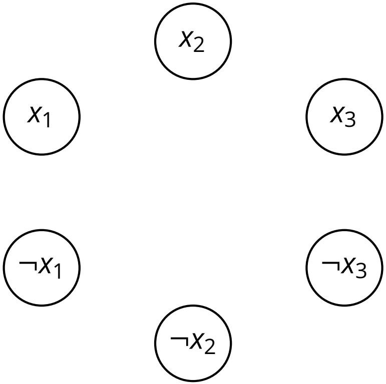

2-CNF-SAT to Graph Reachability

$$\boxed{(x_1 \vee \neg x_3)} \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2)$$

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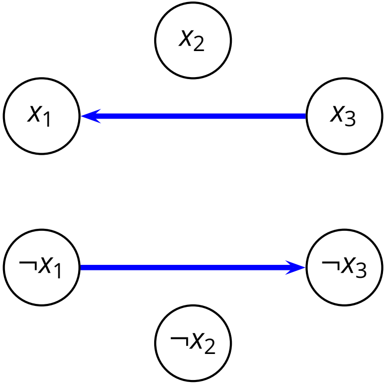

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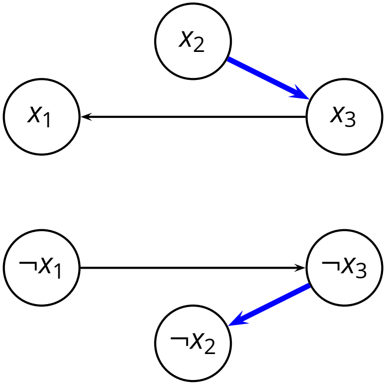

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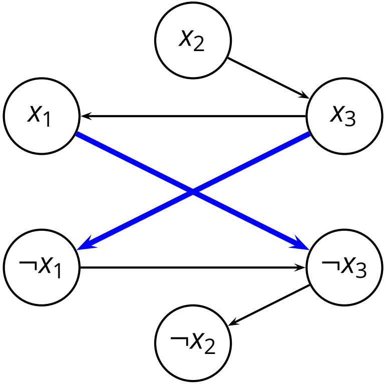

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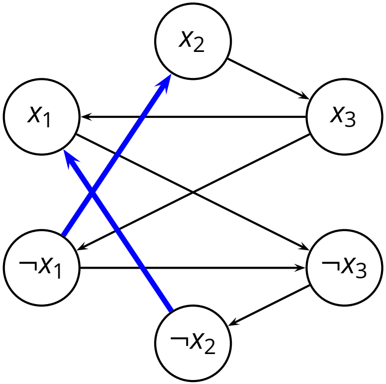

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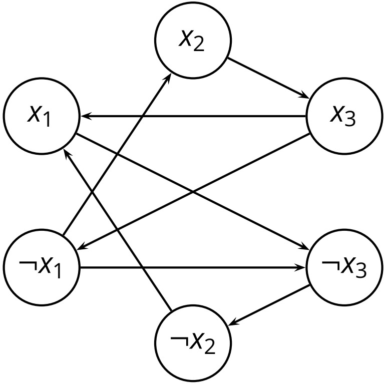

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if and only if
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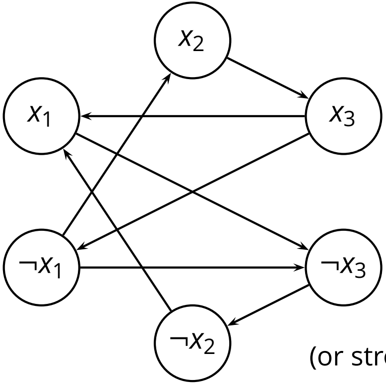

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depth-first search

(or strongly connected components)

Reduction of 2-CNF-SAT

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- 2-CNF-SAT $\in$ *P*

instance of 2-CNF-SAT $\longrightarrow$ $\boxed{?}$ $\cdots\rightarrow$ *solution*

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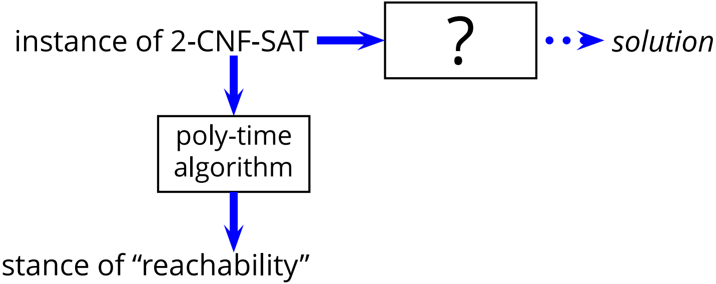

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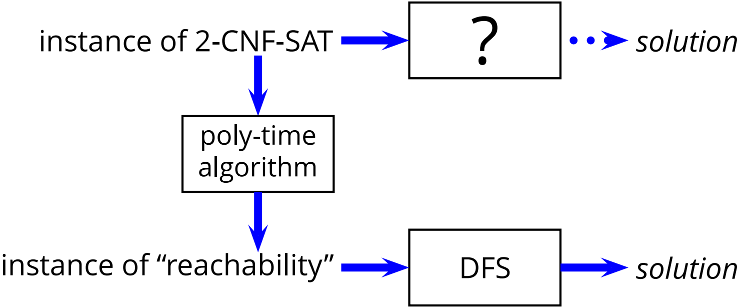

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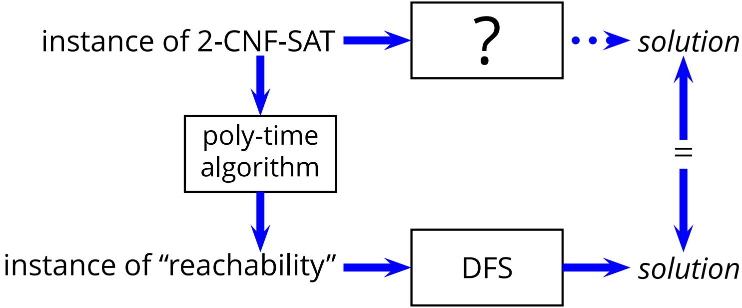
instance of 2-CNF-SAT
?
solution
poly-time algorithm
=
instance of “reachability”
DFS
solution

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- If Q' is NP-hard and *polynomial-time reducible* to Q'', then Q'' is NP-hard
- If Q' is NP-hard and *polynomial-time solvable*, then $\mathrm{P} = \mathrm{NP}$
  - i.e., most researchers believe that there is no such Q'

The First NP-Complete Problem

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- Is there any NP-complete problem?

any problem $Q \in NP \longrightarrow \boxed{\text{polynomial-time reduction}} \longrightarrow$??

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any problem $Q \in NP$ $\rightarrow$ $\boxed{\text{polynomial-time reduction}}$ $\rightarrow$ SAT

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any problem $Q \in NP$ → polynomial-time reduction → SAT

- *Circuit satisfiability (SAT)* was the first problem that was proved NP-hard and, since SAT $\in$ NP, also NP-complete

- Many other problems were then proved NP-complete through polynomial reductions
  - e.g., SAT is polynomial-time reducible to the *longest path* problem
  - therefore, the *longest path* problem is also NP-complete