Divide-and-Conquer Algorithms

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Outline

- Merging (or set union)
- Searching
- Sorting
- Multiplying
- Computing the *median*

Merging (Set Union)

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- **Input:** sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

- **Input:** sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

  - every element of A appears once in X
  - every element of B appears once in X
  - every element of X appears in A or in B or in both

- **Input:** sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

  - every element of A appears once in X
  - every element of B appears once in X
  - every element of X appears in A or in B or in both

- **Example:**

  $A = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10 \rangle$

  $B = \langle 51, 21, 14, 15, 27, 31, 2 \rangle$

  $X =$

- **Input:** sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

  - every element of A appears once in X
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- **Example:**

  $A = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10 \rangle$

  $B = \langle 51, 21, 14, 15, 27, 31, 2 \rangle$

  $X = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10, 15, 27, 2 \rangle$

A Simple Merge Algorithm

- Algorithm strategy

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  - iterate through every position i, first through A, and then B
  - output a_i if a_i is not in $\langle a_1, a_2, \ldots, a_{i-1} \rangle$
  - output b_i if b_i is not in $\langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_{i-1} \rangle$

A Simple Merge Algorithm

- Algorithm strategy
  - iterate through every position i, first through A, and then B
  - output a_i if a_i is not in $\langle a_1, a_2, \ldots, a_{i-1} \rangle$
  - output b_i if b_i is not in $\langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_{i-1} \rangle$

```
MergeSimple(A, B)
for i = 1 to length(A)
    if not Find(A[1 .. i − 1], A[i])
        output A[i]
for i = 1 to length(B)
    if not Find(A, B[i]) and not Find(B[1 .. i − 1], B[i])
        output B[i]
```

Complexity

```
MergeSimple(A, B)
for i = 1 to length(A)
        if not Find(A[1 . . i − 1], A[i])
                output A[i]
for i = 1 to length(B)
        if not Find(A, B[i]) and not Find(B[1 . . i − 1], B[i])
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Complexity

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for i = 1 to length(A)
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    if not Find(A, B[i]) and not Find(B[1 . . i − 1], B[i])
        output B[i]
```

let $n = length(A) + length(B)$

$$T(n) = \sum_{i=1}^{length(A)} T_{\textsc{Find}}(i) + \sum_{i=1}^{length(B)} \left(T_{\textsc{Find}}(i) + T_{\textsc{Find}}(length(A))\right)$$

Complexity

```
MERGESIMPLE(A, B)
for i = 1 to length(A)
    if not FIND(A[1 . . i − 1], A[i])
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```

let $n = length(A) + length(B)$

$$T(n) = \sum_{i=1}^{length(A)} T_{\text{FIND}}(i) + \sum_{i=1}^{length(B)} \left(T_{\text{FIND}}(i) + T_{\text{FIND}}(length(A))\right)$$

$$T(n) = \sum_{i=1}^{n} T_{\text{FIND}}(i)$$

- **Input:** a sequence *A* and a value *key*

  Output: TRUE if *A* contains *key*, or FALSE otherwise

Searching

- **Input:** a sequence A and a value *key*

  Output: TRUE if A contains *key*, or FALSE otherwise

```
FIND(A, key)
for i = 1 to length(A)
    if A[i] == key
        return TRUE
return FALSE
```

```
FIND(A, begin, end, key)
for i = begin to end
    if A[i] == key
        return TRUE
return FALSE
```

- **Input:** a sequence A and a value *key*

  Output: TRUE if A contains *key*, or FALSE otherwise

```
FIND(A, key)
for i = 1 to length(A)
    if A[i] == key
        return TRUE
return FALSE
```

```
FIND(A, begin, end, key)
for i = begin to end
    if A[i] == key
        return TRUE
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```

- The complexity of **Find** is

Searching

- **Input:** a sequence A and a value *key*

  Output: TRUE if A contains *key*, or FALSE otherwise

```
FIND(A, key)
  for i = 1 to length(A)
      if A[i] == key
          return TRUE
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```

```
FIND(A, begin, end, key)
  for i = begin to end
      if A[i] == key
          return TRUE
  return FALSE
```

- The complexity of **FIND** is

$$T(n) = O(n)$$

- **Input:** a sequence A and a value *key*

  Output: TRUE if A contains *key*, or FALSE otherwise

Searching in a List

- **Input:** a sequence *A* and a value *key*

  Output: TRUE if *A* contains *key*, or FALSE otherwise

```
FINDINLIST(A, key)
item = first(A)
while item ≠ last(A)
        if value(item) == key
                return TRUE
        item = next(item)
return FALSE
```

Searching in a List

- **Input:** a sequence *A* and a value *key*

  Output: TRUE if *A* contains *key*, or FALSE otherwise

```
FindInList(A, key)
item = first(A)
while item ≠ last(A)
        if value(item) == key
                return TRUE
        item = next(item)
return FALSE
```

- The complexity of **FindInList** is

Searching in a List

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  Output: TRUE if *A* contains *key*, or FALSE otherwise

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```

- The complexity of **FINDINLIST** is

$$T(n) = O(n)$$

Complexity of MergeSimple

```
MergeSimple(A, B)
for i = 1 to length(A)
        if not Find(A[1 . . i − 1], A[i])
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for i = 1 to length(B)
        if not Find(A, B[i]) and not Find(B[1 . . i − 1], B[i])
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Complexity of MergeSimple

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```

$$T(n) = \sum_{i=1}^{n} T_{\textsc{Find}}(i)$$

Complexity of MergeSimple

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for i = 1 to length(B)
        if not Find(A, B[i]) and not Find(B[1 . . i − 1], B[i])
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```

$$T(n) = \sum_{i=1}^{n} T_{\textsc{Find}}(i)$$

$$T(n) = \sum_{i=1}^{n} O(i) =$$

Complexity of MergeSimple

```
MERGESIMPLE(A, B)
for i = 1 to length(A)
        if not FIND(A[1 . . i − 1], A[i])
                output A[i]
for i = 1 to length(B)
        if not FIND(A, B[i]) and not FIND(B[1 . . i − 1], B[i])
                output B[i]
```

$$T(n) = \sum_{i=1}^{n} T_{\textsc{Find}}(i)$$

$$T(n) = \sum_{i=1}^{n} O(i) = O\left(\frac{n(n+1)}{2}\right) =$$

Complexity of MergeSimple

```
MergeSimple(A, B)
for i = 1 to length(A)
        if not Find(A[1 . . i − 1], A[i])
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for i = 1 to length(B)
        if not Find(A, B[i]) and not Find(B[1 . . i − 1], B[i])
                output B[i]
```

$$T(n) = \sum_{i=1}^{n} T_{\textsc{Find}}(i)$$

$$T(n) = \sum_{i=1}^{n} O(i) = O\left(\frac{n(n+1)}{2}\right) = O(n^2)$$

- **Input:** a *sorted* sequence *A* and a value *key*

  Output: TRUE if *A* contains *key*, or FALSE otherwise

Searching (2)

- **Input:** a *sorted* sequence A and a value *key*

  Output: TRUE if A contains *key*, or FALSE otherwise

```
BINARYSEARCH(A, key)
first = 1
last = length(A)
while first ≤ last
        middle = ⌈(first + last)/2⌉
        if A[middle] == key
                return TRUE
        elseif first = last
                return FALSE
        elseif A[middle] > key
                last = middle − 1
        else first = middle + 1
return FALSE
```

Binary Search

```
BinarySearch(A, key)
first = 1
last = length(A)
while first ≤ last
    middle = ⌈(first + last)/2⌉
    if A[middle] == key
        return TRUE
    elseif first = last
        return FALSE
    elseif A[middle] > key
        last = middle − 1
    else first = middle + 1
return FALSE
```

Binary Search

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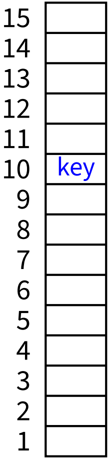
return FALSE
```



Binary Search

```
BINARYSEARCH(A, key)
first = 1
last = length(A)
while first ≤ last
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        if A[middle] == key
                return TRUE
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                return FALSE
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return FALSE
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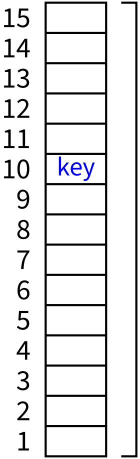
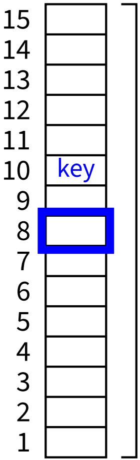


Binary Search

BinarySearch(*A*, *key*)

```
first = 1
last = length(A)
while first ≤ last
    middle = ⌈(first + last)/2⌉
    if A[middle] == key
        return TRUE
    elseif first = last
        return FALSE
    elseif A[middle] > key
        last = middle − 1
    else first = middle + 1
return FALSE
```



Binary Search

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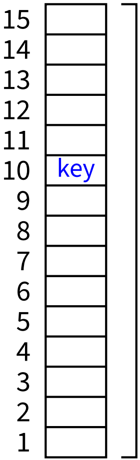

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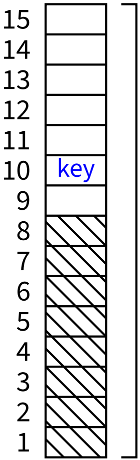

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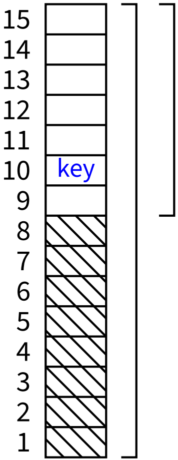

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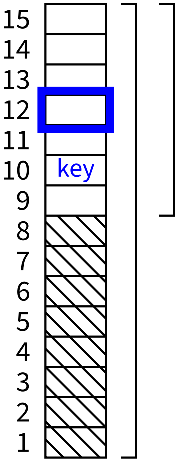

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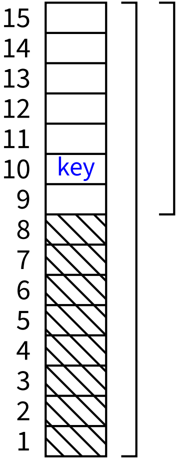

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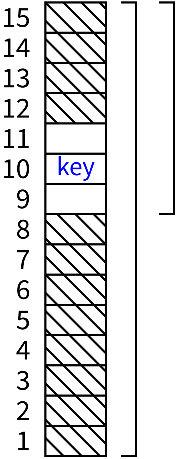

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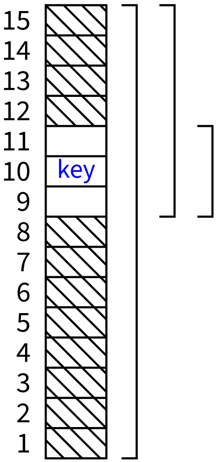

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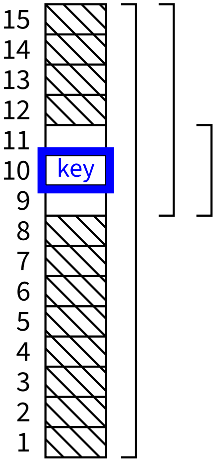

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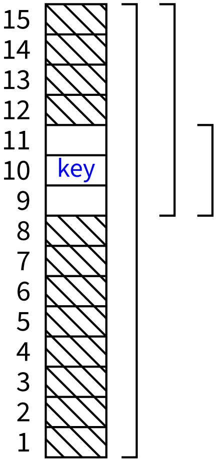

Binary Search

```
BinarySearch(A, key)
first = 1
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while first ≤ last
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    elseif first = last
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```

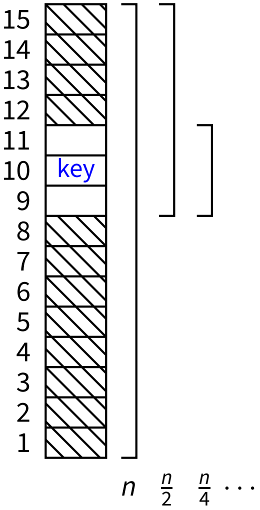

```
BINARYSEARCH(A, key)
first = 1
last = length(A)
while first ≤ last
    middle = ⌈(first + last)/2⌉
    if A[middle] == key
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return FALSE
```
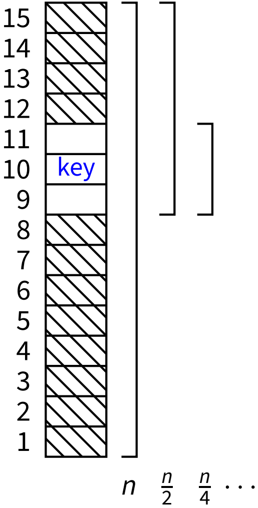



$$T(n) = O(\log n)$$

- A slightly different problem:

  Input: two *sorted* sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$, where $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_m$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

  - every element of A appears once in X
  - every element of B appears once in X
  - every element of X appears in A or in B or in both

A Better Merge Algorithm

```
MergeSimple2(A, B)
for i = 1 to length(A)
    if not BinarySearch(A[1 . . i − 1], A[i])
        output A[i]
for i = 1 to length(B)
    if not BinarySearch(A, B[i])
    and not BinarySearch(B[1 . . i − 1], B[i])
        output B[i]
```

A Better Merge Algorithm

```
MERGESIMPLE2(A, B)
for i = 1 to length(A)
    if not BINARYSEARCH(A[1 . . i − 1], A[i])
        output A[i]
for i = 1 to length(B)
    if not BINARYSEARCH(A, B[i])
    and not BINARYSEARCH(B[1 . . i − 1], B[i])
        output B[i]
```

$$T(n) = \sum_{i=1}^{n} O(\log i) =$$

A Better Merge Algorithm

```
MergeSimple2(A, B)
for i = 1 to length(A)
    if not BinarySearch(A[1 .. i − 1], A[i])
        output A[i]
for i = 1 to length(B)
    if not BinarySearch(A, B[i])
    and not BinarySearch(B[1 .. i − 1], B[i])
        output B[i]
```

$$T(n) = \sum_{i=1}^{n} O(\log i) = O(n \log n)$$

A Better Merge Algorithm

```
MERGESIMPLE2(A, B)
for i = 1 to length(A)
    if not BINARYSEARCH(A[1 . . i − 1], A[i])
        output A[i]
for i = 1 to length(B)
    if not BINARYSEARCH(A, B[i])
    and not BINARYSEARCH(B[1 . . i − 1], B[i])
        output B[i]
```

$$T(n) = \sum_{i=1}^{n} O(\log i) = O(n \log n)$$

Better than $O(n^2)$, but can we do even better than $O(n \log n)$?

An Even Better Merge Algorithm

- *Intuition:* A and B are *sorted*
  e.g.
  $A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$
  $B = \langle 1, 5, 6, 7, 34, 35, 40, 41, 43 \rangle$

- *Intuition: A* and B are *sorted*
  e.g.
  $A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$
  $B = \langle 1, 5, 6, 7, 34, 35, 40, 41, 43 \rangle$
  so just like in **BinarySearch** I can avoid looking for an element x if the *first* element I see is $y > x$

An Even Better Merge Algorithm

- *Intuition:* A and B are *sorted*
  e.g.
  $A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$
  $B = \langle 1, 5, 6, 7, 34, 35, 40, 41, 43 \rangle$
  so just like in **BinarySearch** I can avoid looking for an element x if the *first* element I see is $y > x$

- High-level algorithm strategy
  - step through every position i of A and every position j of B
  - output a_i and advance i if $a_i \leq b_j$ or if j is beyond the end of B
  - output b_j and advance j if $a_i \geq b_j$ or if i is beyond the end of A

MERGE Algorithm

A	3	7	12	13	34	37	70	75	80
B	1	5	6	7	34	35	40	41	43

$i = 1$

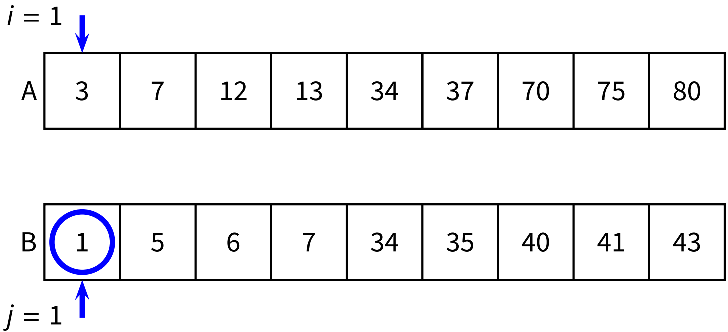
A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 1$

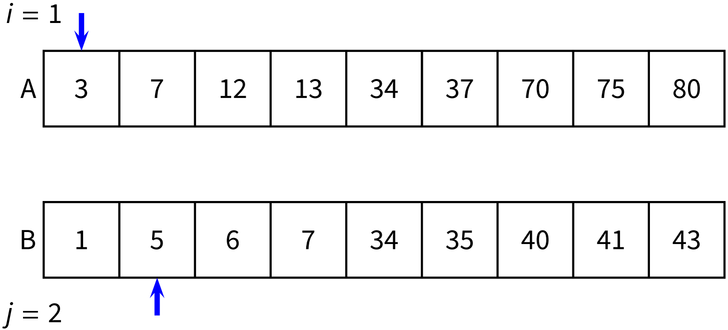
Output:

Merge Algorithm

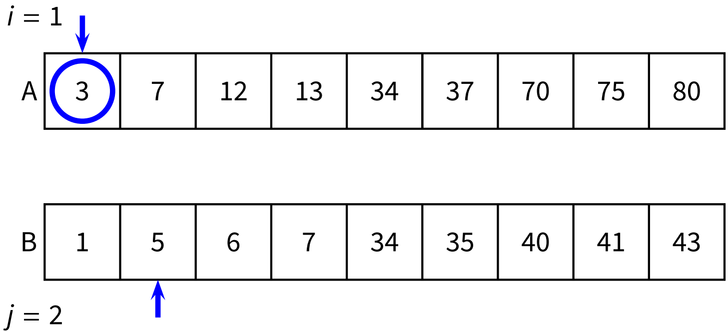


Output:

MERGE Algorithm



Output: 1



Output: 1

MERGE Algorithm

$i = 2$

A	3	7	12	13	34	37	70	75	80

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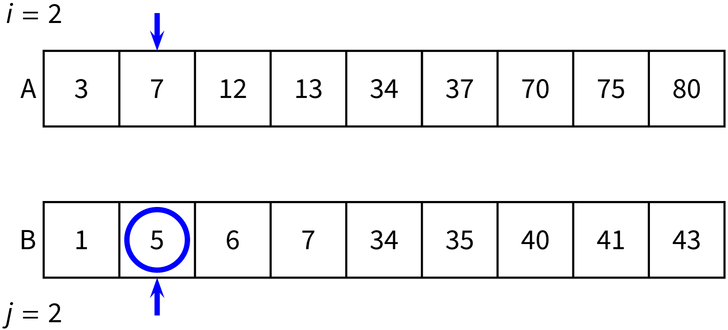
$j = 2$

Output: 1 3

MERGE Algorithm



Output: 1 3

MERGE Algorithm

$i = 2$

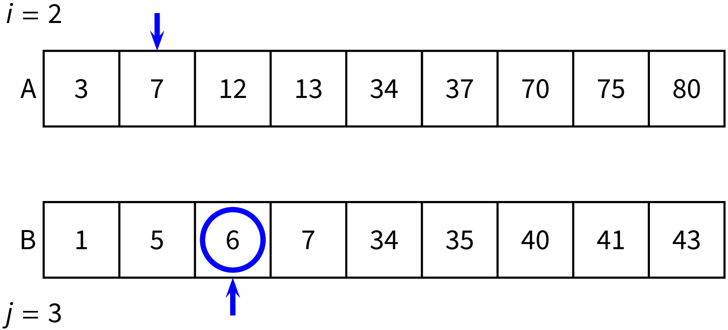
A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 3$

Output: 1 3 5

Merge Algorithm



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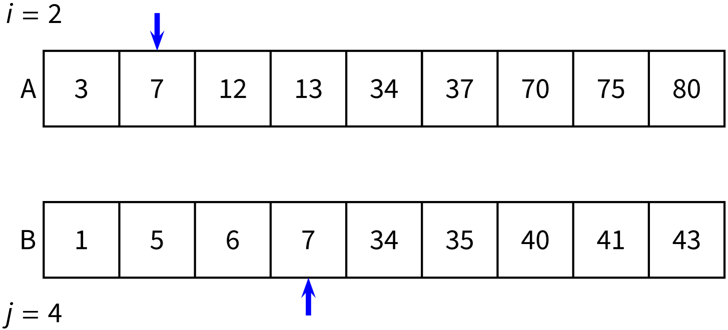

Output: 1 3 5 6

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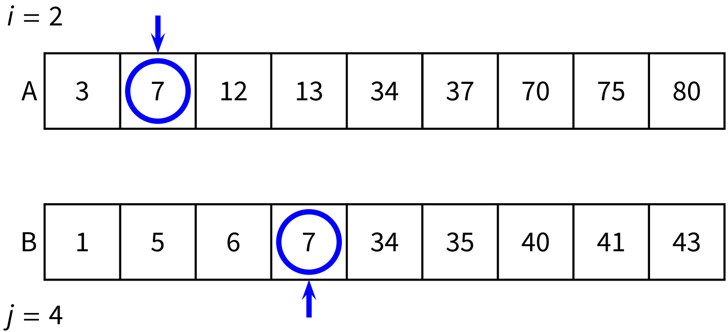

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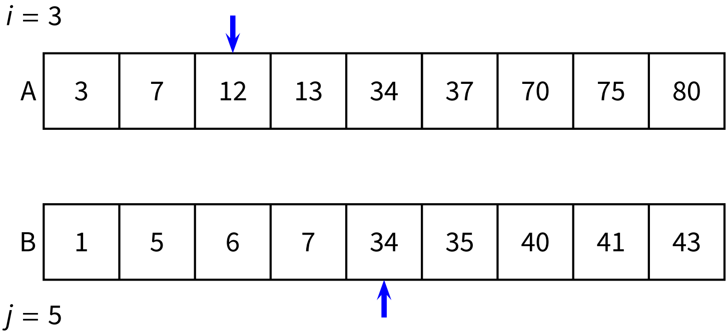

Output: 1 3 5 6 7

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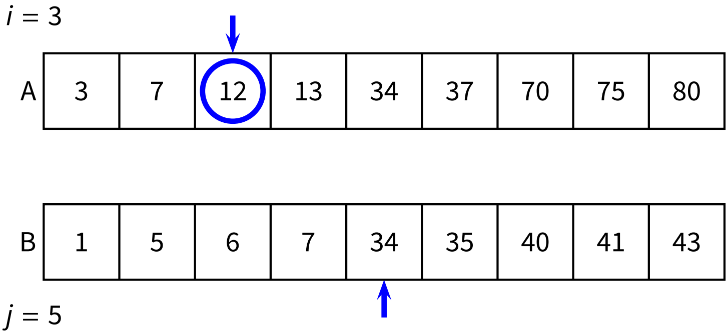

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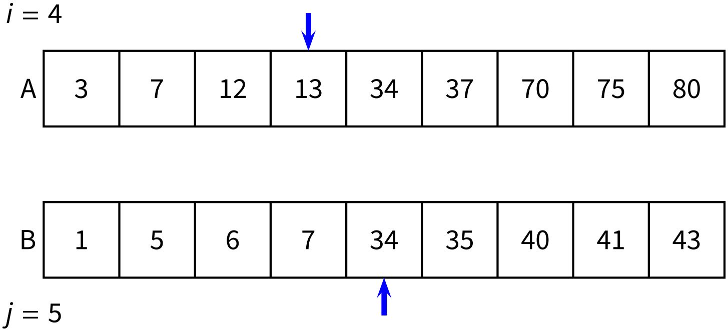
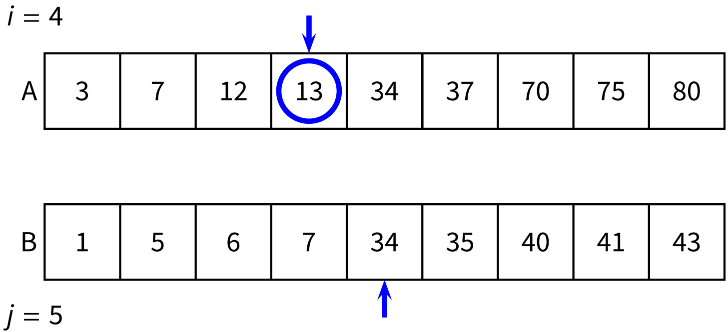


Output: 1 3 5 6 7 12



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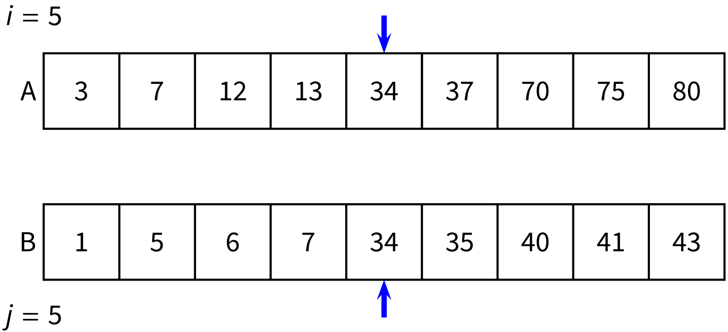

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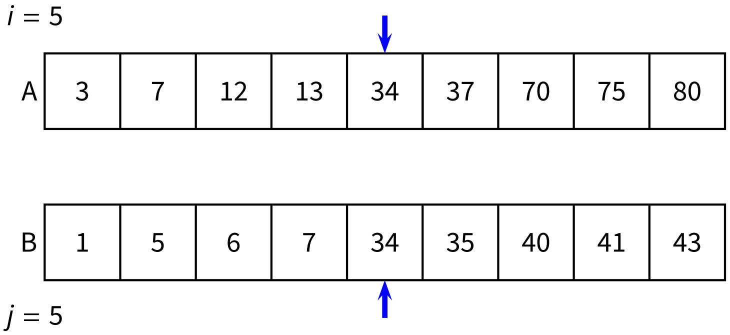

Output: 1 3 5 6 7 12 13...

Merge Algorithm (2)

```
MERGE(A, B)
i, j = 1
X = ∅
while i ≤ length(A) or j ≤ length(B)
    if i > length(A)
        X = X ∘ B[j]          // appends B[j] to X
        j = j + 1
    elseif j > length(B)
        X = X ∘ A[i]
        i = i + 1
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- This algorithm is incorrect! (Exercise: fix it)

Complexity of MERGE

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$$T(n) = \Theta(n)$$

- Can we do better? No!
  - we have to output $n = \mathit{length}(A) + \mathit{length}(B)$ elements

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  - merges two *sorted* sequences
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  - use a variant of **Merge** that outputs *all* elements of its input sequences
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  - this suggests a recursive algorithm

Merge Sort

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MERGESORT(A)
if length(A) == 1
        return A
m = ⌊length(A)/2⌋
A_L = MERGESORT(A[1 .. m])
A_R = MERGESORT(A[m + 1 .. length(A)])
return MERGE(A_L, A_R)
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- Complexity analysis

$$T(n) = T_{\text{divide}} + \sum_{i=1}^{k} T(|A_i|) + T_{\text{combine}}$$

we might analyze this formula another time…

A Divide-and-Conquer Merge

```
MERGER(A, B)
    if length(A) == 0
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    if length(B) == 0
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    if A[1] < B[1]
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- Can we do better?

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- Can we do better? No! (We knew that already)

Divide-and-Conquer Multiplication

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  which means $x = 2^{\ell/2}x_L + x_R$ and $y = 2^{\ell/2}y_L + y_R$, so…

$$\begin{aligned} xy &= (2^{\ell/2}x_L + x_R)(2^{\ell/2}y_L + y_R) \\ &= 2^{\ell}x_Ly_L + 2^{\ell/2}(x_Ly_R + x_Ry_L) + x_Ry_R \end{aligned}$$

  we reduced the problem of multiplying two numbers of ℓ bits into the problem of multiplying *four* numbers of $\ell/2$ bits…

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Only 3 multiplications: $x_L y_L$, $(x_L + x_R)(y_R + y_L)$, and $x_R y_R$

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$$T(\ell) = 3T(\ell/2) + O(\ell)$$

which, as we will see, leads to a much better complexity

$$T(\ell) = O(\ell^{\log_2 3}) = O(\ell^{1.59})$$

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- Is it correct? Yes

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- Can we do better? Intuitively, yes...

Computing the Median (2)

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    the 6th smallest element of A—a.k.a. $select(A, 6)$—is 8

k-Smallest Element

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It is the 2nd smallest value of A_R

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We use $select(A, k)$ to denote the k-smallest element of A

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- We pick *a random element of A*

Selection Algorithm

```
SELECTION(A, k)
  v = A[random(1 ... |A|)]
  A_L, A_v, A_R = ∅
  for i = 1 to |A|
      if A[i] < v
          A_L = A_L ∪ A[i]
      elseif A[i] == v
          A_v = A_v ∪ A[i]
      else A_R = A_R ∪ A[i]
  if k ≤ |A_L|
      return SELECTION(A_L, k)
  elseif k > |A_L| + |A_v|
      return SELECTION(A_R, k − |A_L| − |A_v|)
  else return v
```