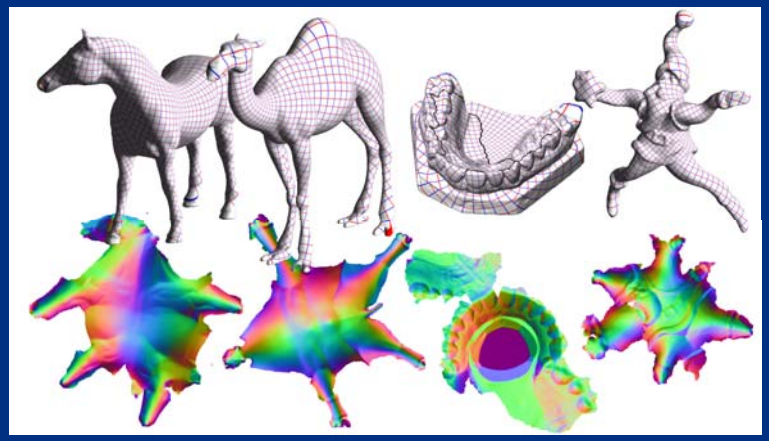
Siggraph Course

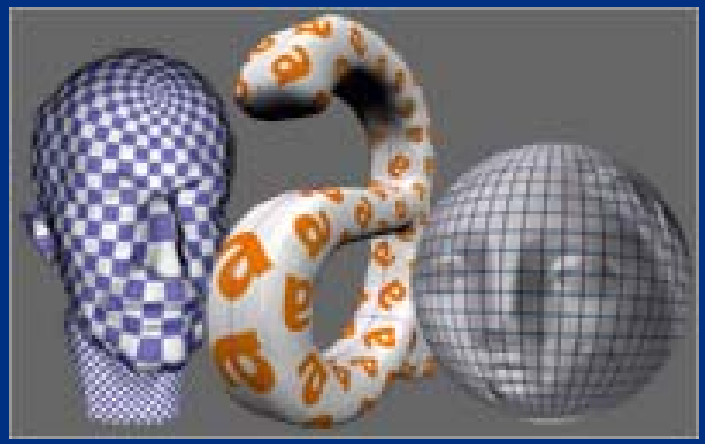
Mesh Parameterization: Theory and Practice

SIGGRAPH2007

Indirect Methods (2D)

Alla Sheffer





Alternative Variables

- Most methods look directly for UV coordinates
- Alternative:
  - Use parameters which define 2D mesh uniquely
  - Search in alternative parameter space & then convert to UV
  - *Enforce constraints defining 2D mesh in parameter space*
- Examples
  - 2D mesh angles [Sheffer & de Sturler:00; Kharevych:06]
  - Gradients [Gu & Yau:03; Ray:06]
  - Angle deficit [Gotsman:07]

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Angle Space

- Triangular 2D mesh is defined by its angles
- Formulate parameterization as problem in angle space [Sheffer & de Sturler,00]
- Angle based formulation:
  - Distortion as function of angles (conformality)
  - Validity: set of angle constraints
  - Convert solution to UV

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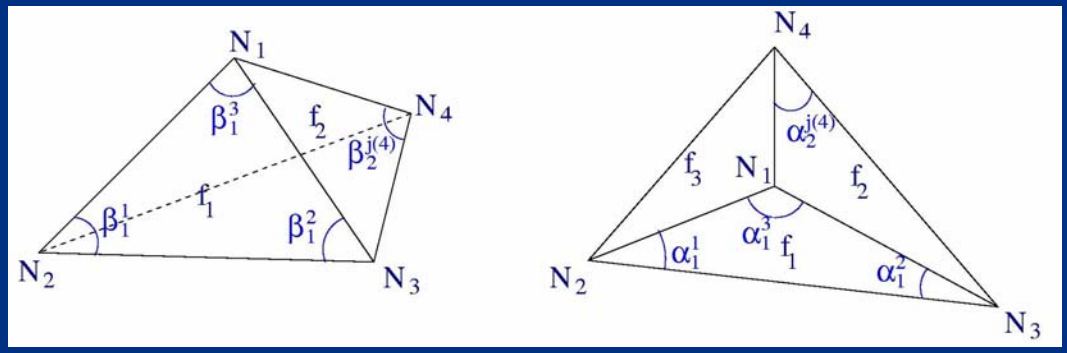
ABF Formulation
[Sheffer & de Sturler:00]



- Distortion:
  - 2D/3D angle difference

$$\sum_{t\in T,\, j=1..3} w_j^t \left(\alpha_j^t - \beta_j^t\right)^2, \; w_j^t = 1 \Big/ \beta_j^{t\,2}$$

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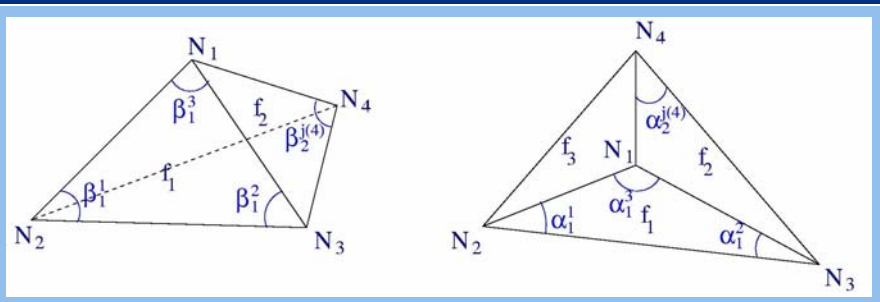
ABF Formulation



- *Constraints:*
  - Triangle validity:

$$\forall t \in T, \quad \alpha_1^t + \alpha_2^t + \alpha_3^t - \pi = 0;$$

  - Planarity:

$$\forall v \in V_{int}, \quad \sum_{(t,k)\in v^*} \alpha_k^t - 2\pi = 0$$

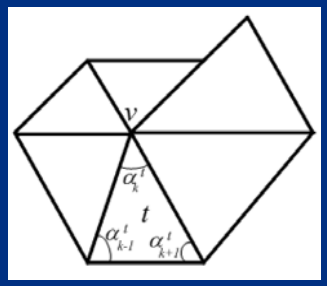
  - Reconstruction

$$\forall v \in V_{int}, \quad \prod_{(t,k)\in v^*} \sin \alpha_{k\oplus 1}^t - \prod_{(t,k)\in v^*} \sin \alpha_{k\ominus 1}^t = 0$$

  - Positivity $\alpha_j^t > 0$

- Distortion:

$$\sum_{t\in T,\, j=1..3} w_j^t \left(\alpha_j^t - \beta_j^t\right)^2, w_j^t = 1/\beta_j^{t\,2}$$



- Solve - constrained optimization (Lagrange multipliers)

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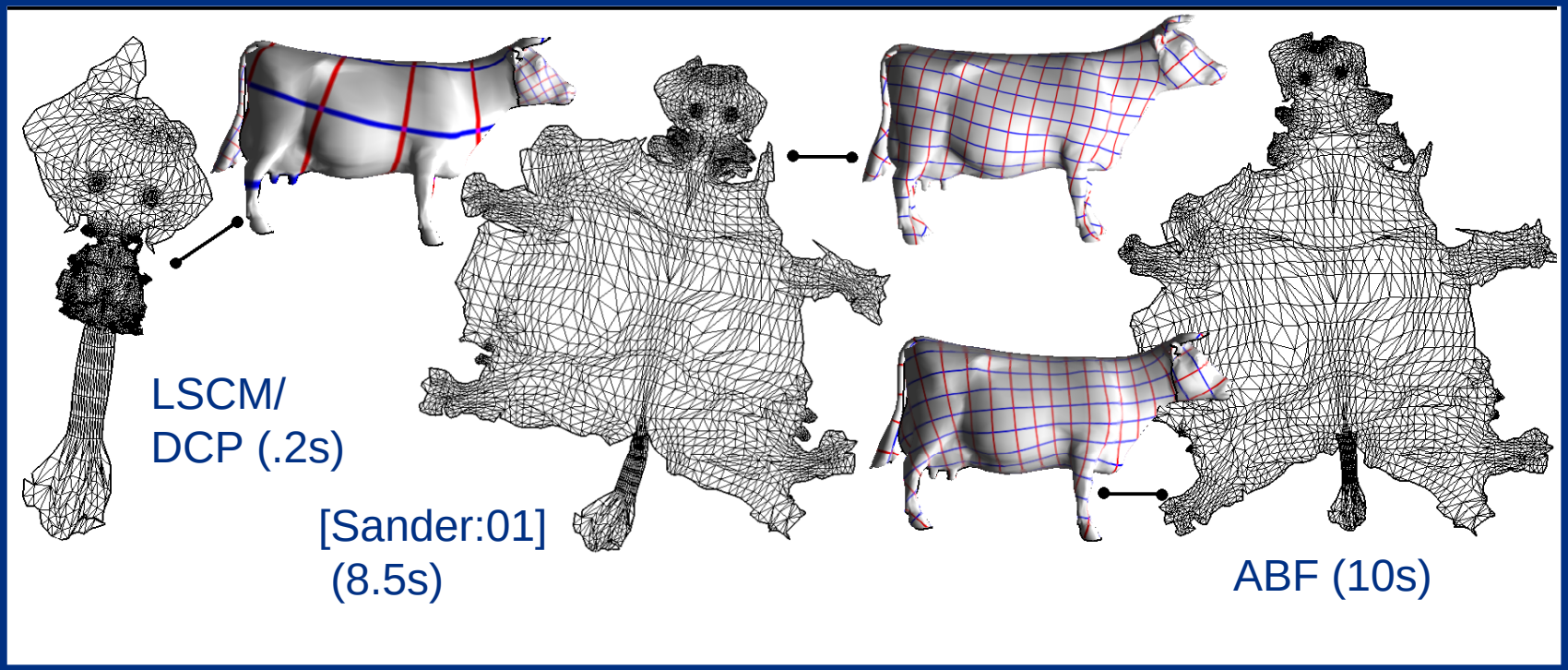
Comparisons



- Validity (no local flips) guaranteed !!!

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ABF++ : Up to x100 speedup
[Sheffer et al:05]

ABF

- Solver:
  - Newton
    - At each step solve

$$\nabla^2 F\delta = -\nabla F, \nabla^2 F = \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}$$

- Conversion
  - Triangle unfolding
    - accumulates error

ABF++

- Solver:
  - Gauss-Newton
    - Allows drastic system simplification

Diagonal

$$\nabla^2 F = \begin{pmatrix} \Lambda & B \\ B^T & 0 \end{pmatrix}$$

- Conversion:
  - LSCM (α_j^t as target angles)
    - allow less accurate solution

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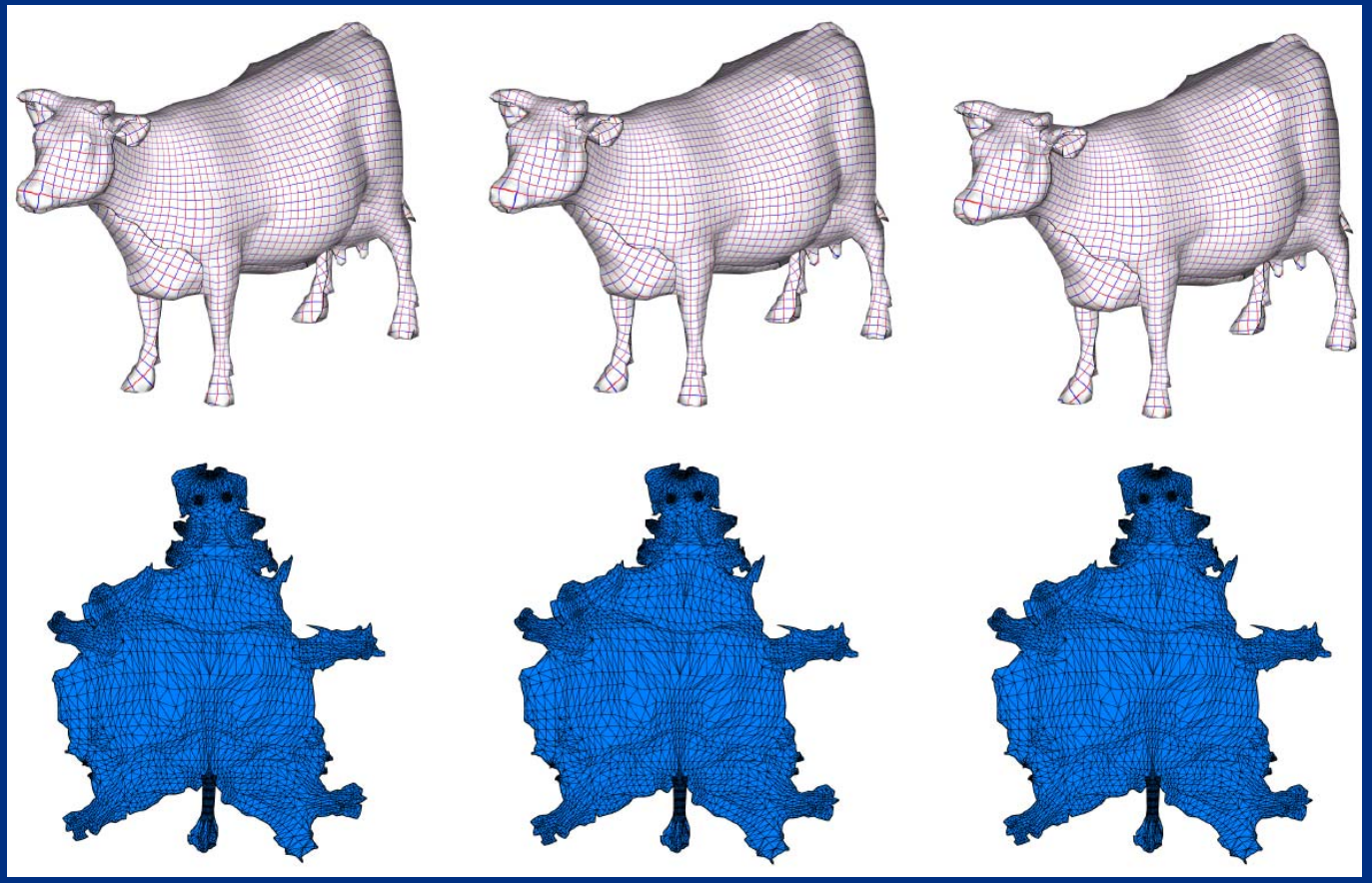
Convergence



1 Iteration

$L_2^{stretch} = 1.10634, \quad L_2^{shear} = 4.69e^{-4}$

2 iterations

$L_2^{stretch} = 1.10641, \quad L_2^{shear} = 4.64e^{-4}$

10 iterations

$L_2^{stretch} = 1.10638, \quad L_2^{shear} = 4.64e^{-4}$

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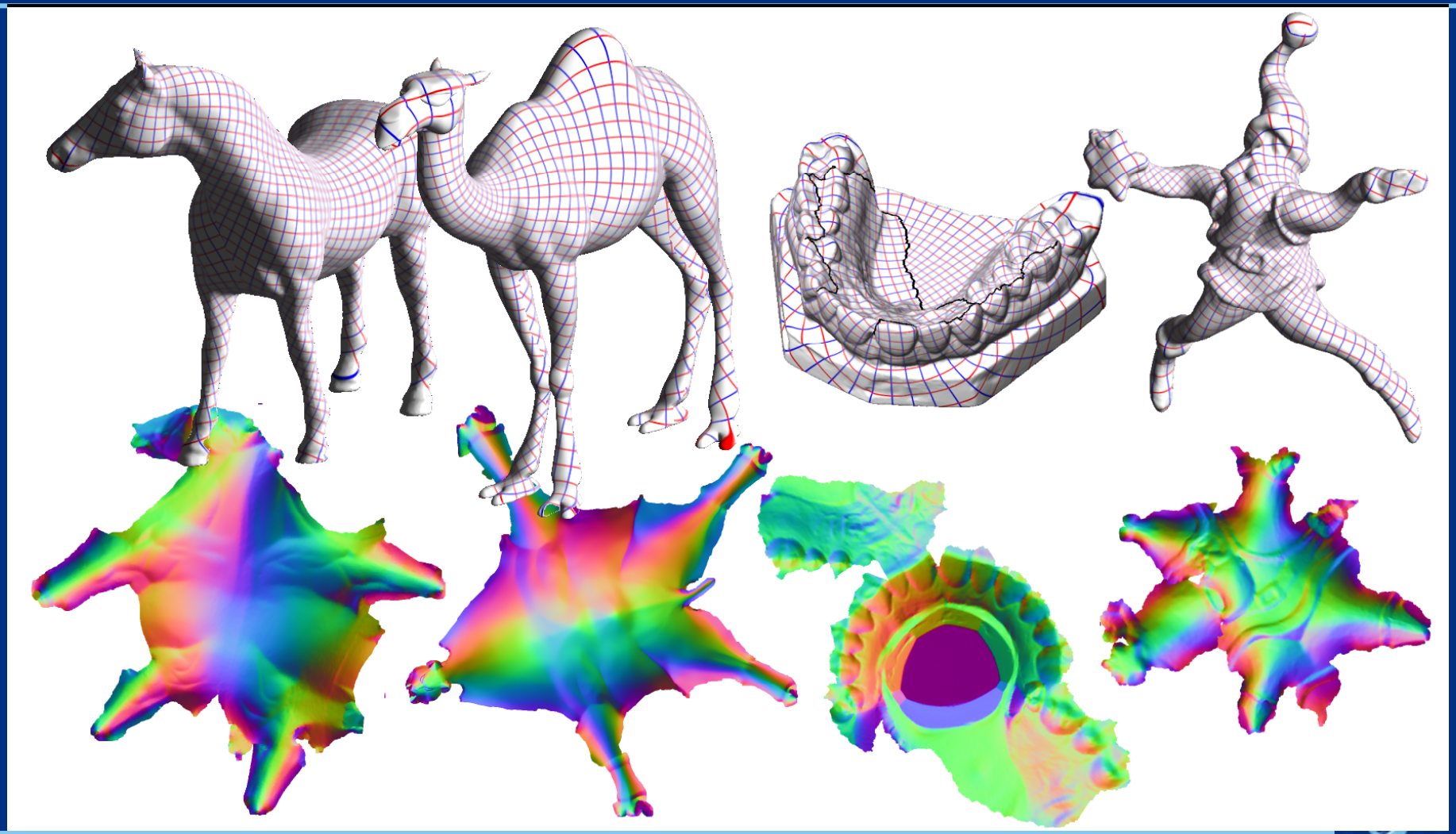
Results



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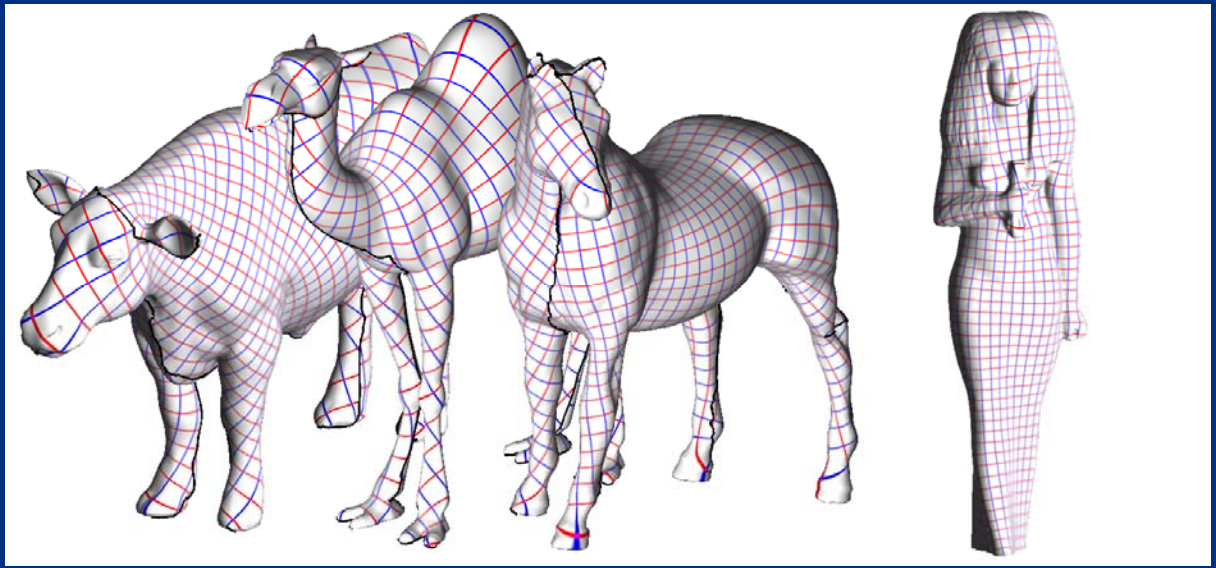
Speedup: ABF vs ABF++



	ABF++	ABF	LSCM
Bull (35K Δ)	11	106	3
Camel (80K Δ)	55	485	8
Horse (100K Δ)	85	3450	15
Isis (400K Δ)	335	NA	58

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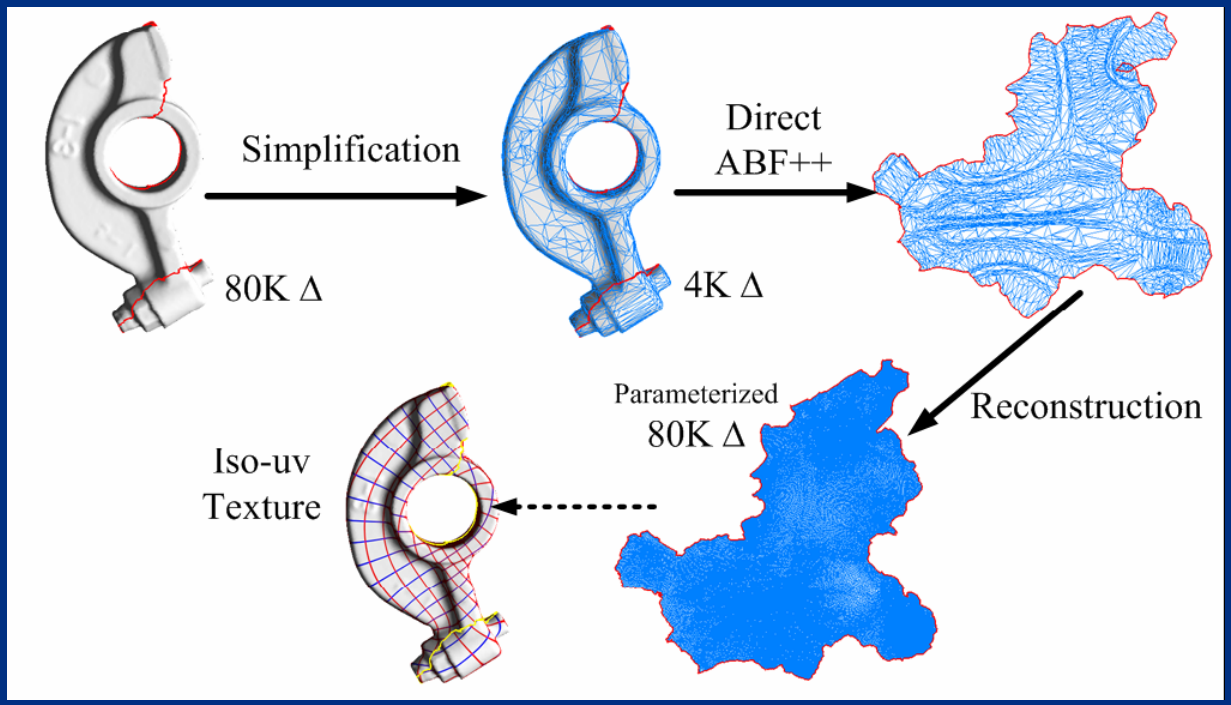
Hierarchical ABF++



- Additional speedup (x10-20) at expense of increased distortion

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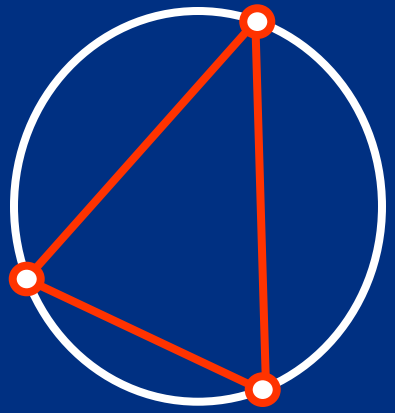
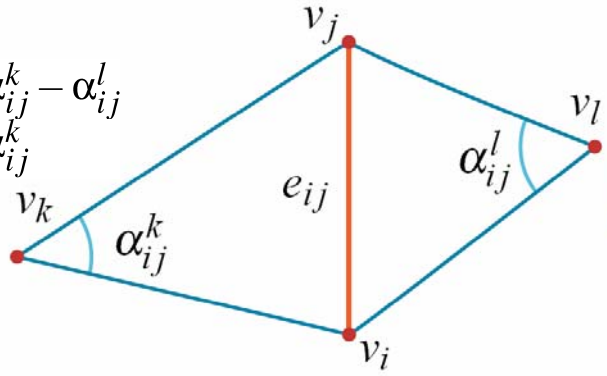
Circle Patterns [Kharevych:06]

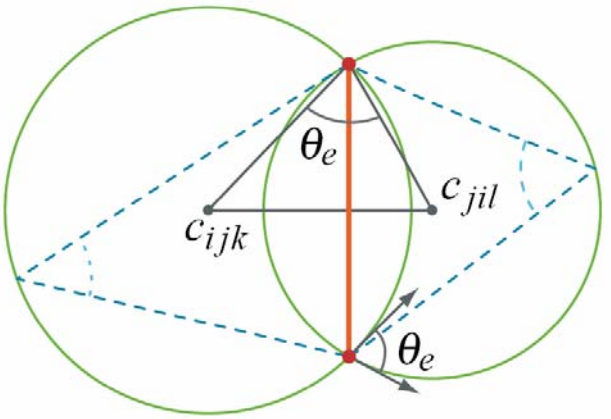
- Three Points make a Triangle… or a Circle
- Local geometry



$$\forall e_{ij} \in E : \theta_e = \begin{cases} \pi - \alpha_{ij}^k - \alpha_{ij}^l \\ \pi - \alpha_{ij}^k \end{cases}$$

Edge angles





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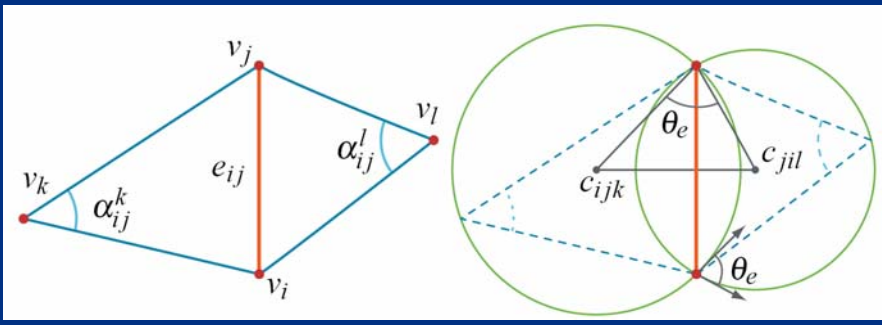
Geometry Preserving Edge Angles



- Edge angle constraints
  - positivity

$$\forall e_{ij} \in E : 0 < \theta_e < \pi$$

  - planarity

$$\forall v_i \in V_{\text{int}} : \sum_{e \ni v_i} \theta_e = 2\pi$$

- Extract from 3D geometry?
- Idea: extract "feasible" triangle angles & convert to edge angles
  - feasible angles close to 3D angles

$$\forall e_{ij} \in E : \theta_e = \begin{cases} \pi - \hat{\alpha}_{ij}^k - \hat{\alpha}_{ij}^l \\ \pi - \hat{\alpha}_{ij}^k \end{cases}$$

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Feasible angles

- Minimize

$$Q(\hat{\alpha}) = \sum |\hat{\alpha}_{ij}^{k} - \beta_{ij}^{k}|^2$$

- Subject to

$$\forall \hat{\alpha}_{ij}^{k} : \hat{\alpha}_{ij}^{k} > 0$$

$$\forall t_{ijk} \in T : \hat{\alpha}_{ij}^{k} + \hat{\alpha}_{jk}^{i} + \hat{\alpha}_{ki}^{j} = \pi$$

$$\forall e_{ij} \in E_{\text{int}} : \hat{\alpha}_{ij}^{k} + \hat{\alpha}_{ij}^{l} < \pi$$

$$\forall v_k \in V_{\text{int}} : \sum_{t_{ijk} \ni v_k} \hat{\alpha}_{ij}^{k} = 2\pi$$

  - Compare to ABF: replace reconstruction constraint

- Solve with quadratic programming
- Convert:

$$\forall e_{ij} \in E : \theta_e = \begin{cases} \pi - \hat{\alpha}_{ij}^{k} - \hat{\alpha}_{ij}^{l} \\ \pi - \hat{\alpha}_{ij}^{k} \end{cases}$$

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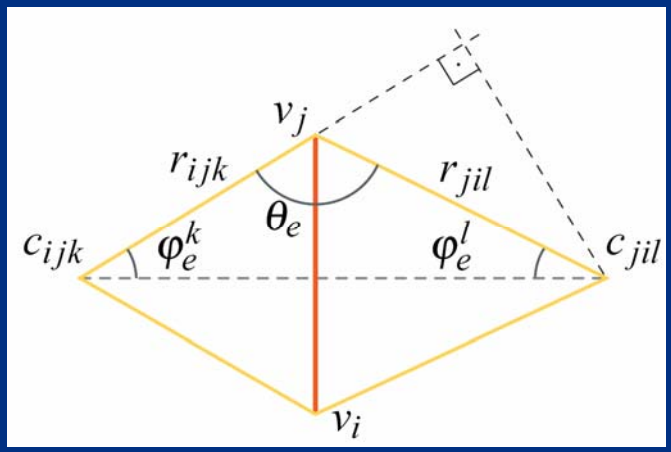
2D Geometry From Edge Angles

- To get radii from edge angles solve global minimization problem

$$\rho_{ijk} = \log r_{ijk}$$

$$S(\rho) = \sum_{e \in E_{\text{int}}} \left(\text{ImLi}(e^{\rho_k - \rho_l + i\theta_e}) + \text{ImLi}(e^{\rho_l - \rho_k + i\theta_e}) - (\pi - \theta_e)(\rho_k + \rho_l) \right) - \sum_{e \in E_{\text{bdy}}} 2(\pi - \theta_e)\rho_k + 2\pi \sum_{t \in T} \rho_t$$



  - Convex energy - Unique minimum
- Given radii and edge angles get UV by unfolding

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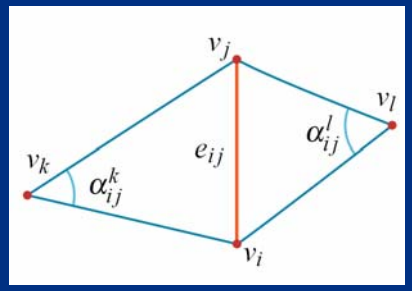
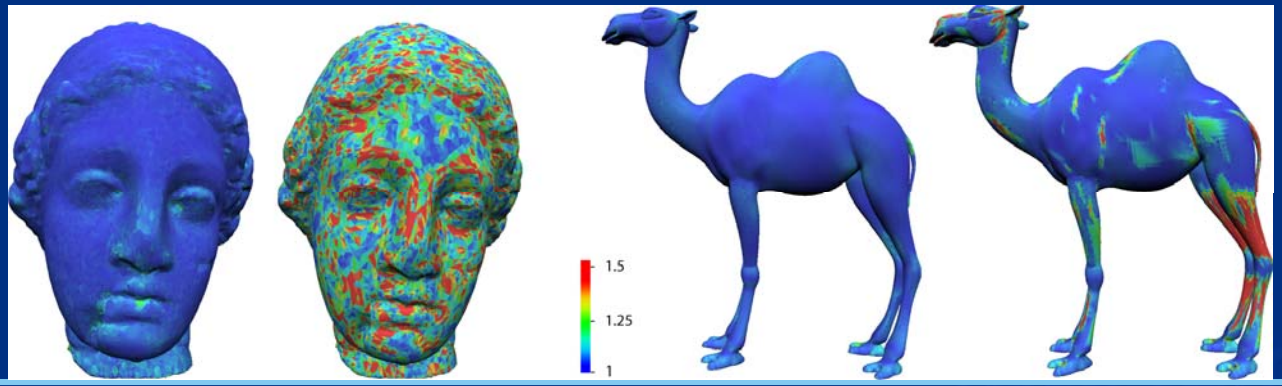
Intrinsic Delaunay

- Enforce: $\forall e_{ij} \in E : \theta_e = \pi - \alpha_{ij}^k - \alpha_{ij}^l$

$$\forall e_{ij} \in E_{\text{int}} : \hat{\alpha}_{ij}^k + \hat{\alpha}_{ij}^l < \pi$$



- Large distortion if 3D mesh not Delaunay
- Solution: Intrinsic Delaunay triangulation
  - perform implicit (local) edge flips in 3D



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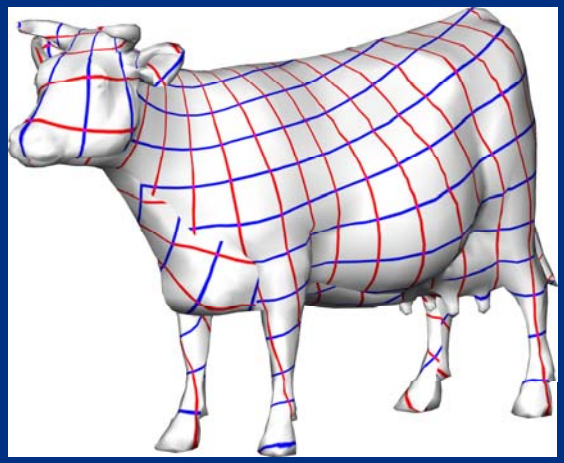
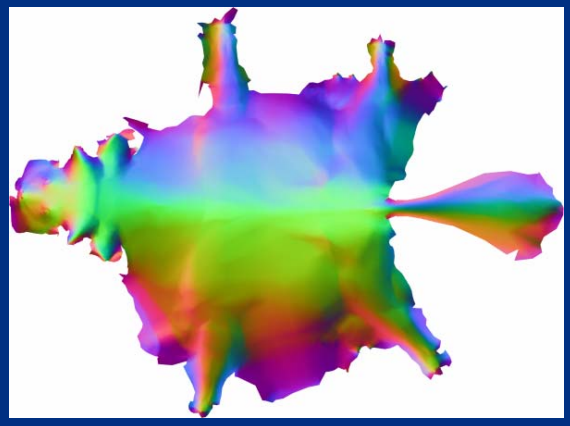
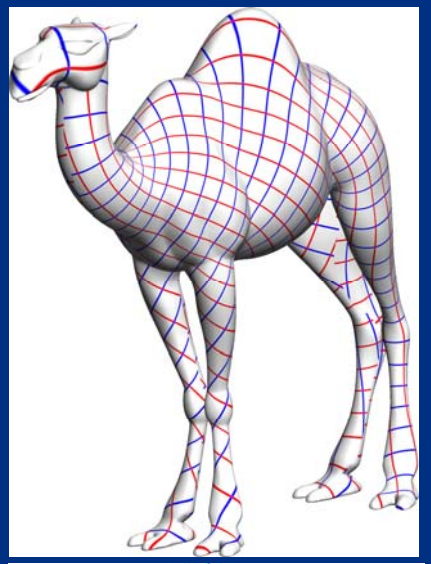
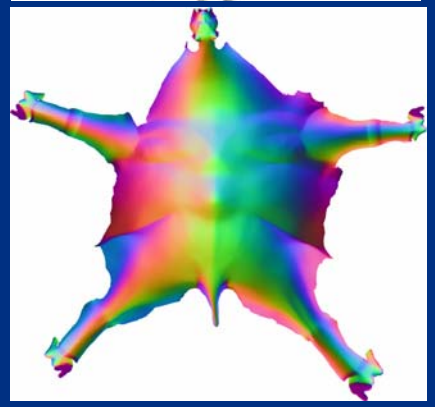
Examples









- Speed comparable to ABF++

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Cone Singularities [Kharevych:06]

- What separates boundary from interior in angle space?
- Answer: Sum of angles at vertex
- Formulation specific
  - Circle patterns
    - Planarity $\forall v_k \in V_{\text{int}} : \sum_{t_{ijk} \ni v_k} \hat{\alpha}_{ij}^k = 2\pi$
  - ABF/ABF++
    - Planarity & Reconstruction

$$\forall v \in V_{int}, \quad \sum_{(t,k)\in v^*} \alpha_k^t - 2\pi = 0$$

$$\forall v \in V_{int}, \quad \prod_{(t,k)\in v^*} \sin \alpha_{k\oplus 1}^t - \prod_{(t,k)\in v^*} \sin \alpha_{k\ominus 1}^t = 0$$

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Cone Singularities

- Idea: Reduce boundary to small set of vertices
- Implementation:
  - Enforce “interior” constraints at all other vertices
- To unfold choose any sequence of edges connecting “boundary” vertices

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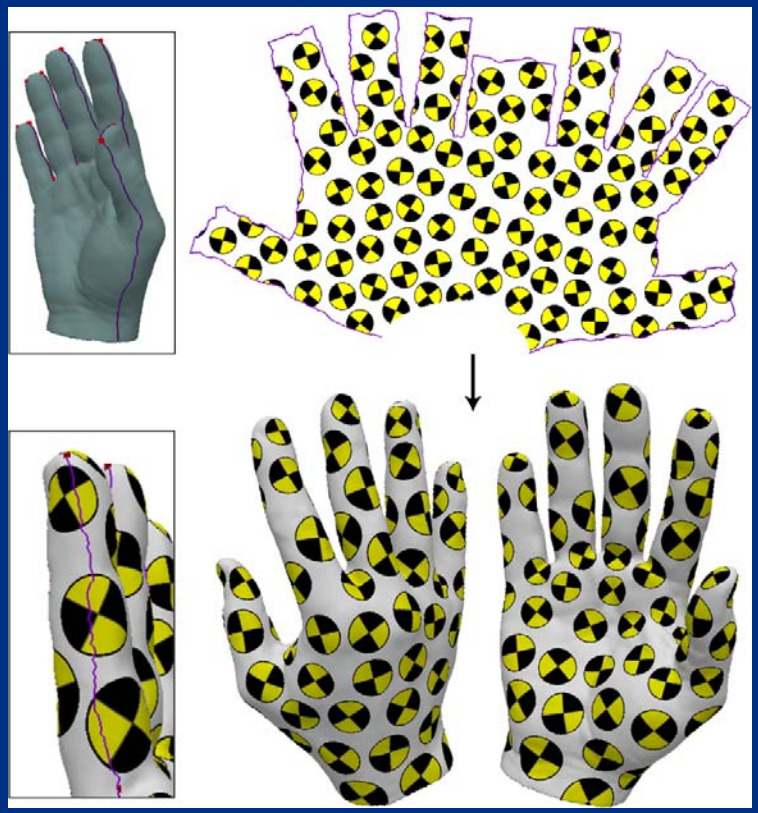
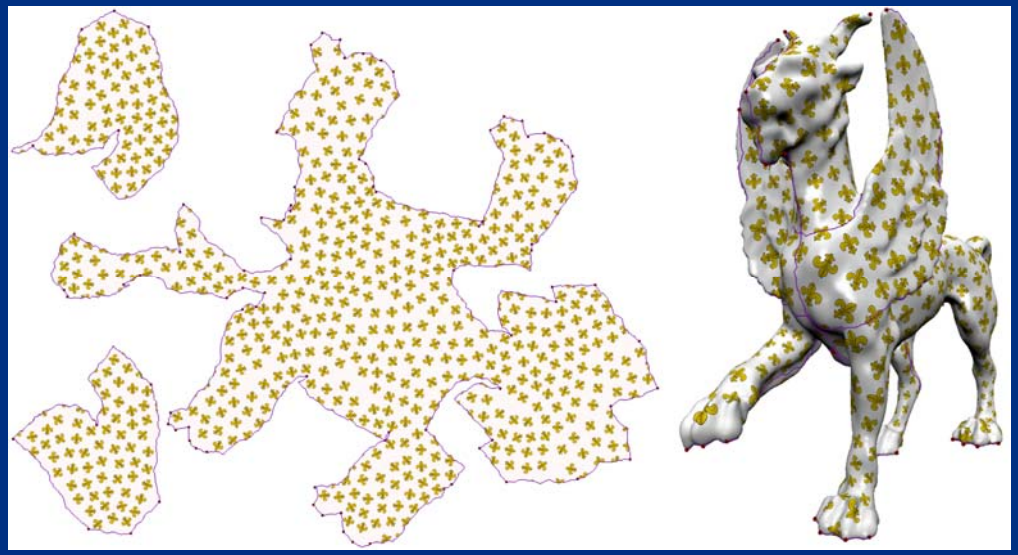
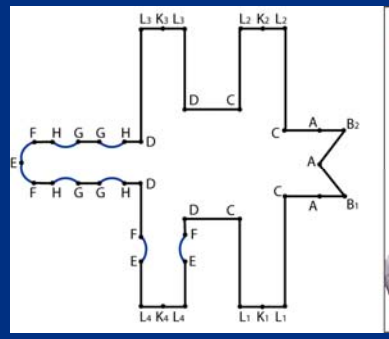
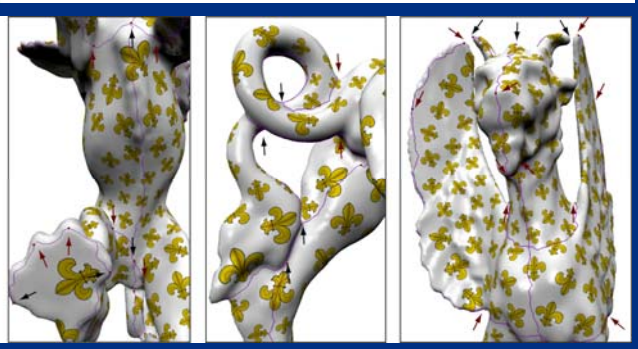
Circle Patterns + Cone Singularities









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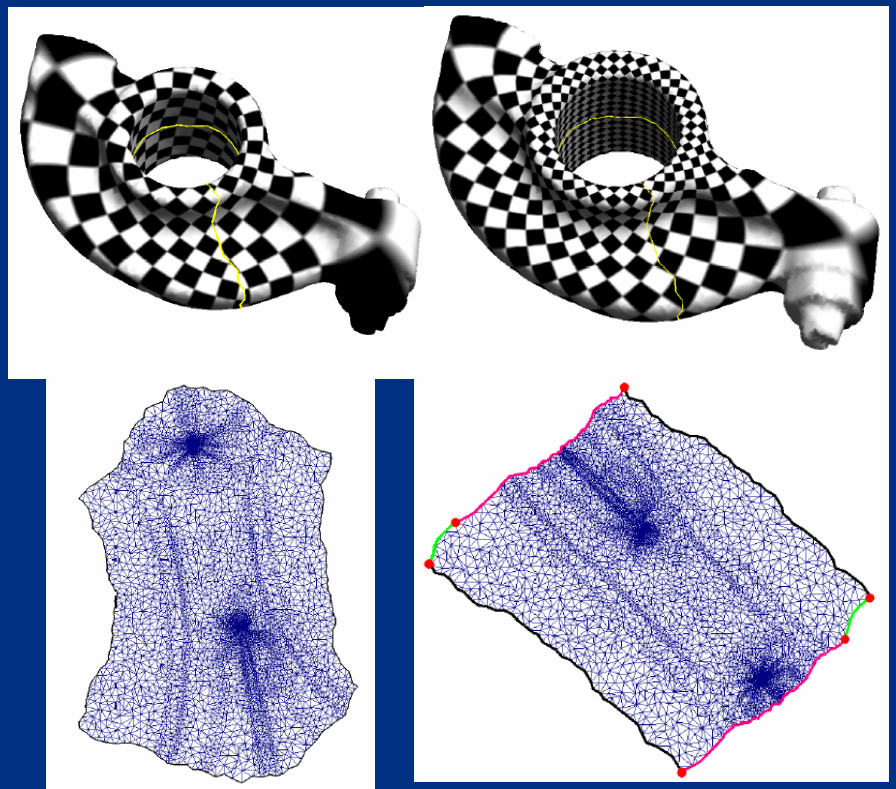
ABF + Cone Singularities



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Recent Advances

- [Zayer:07] Reformulate ABF to increase convergence (1 iter + LSCM)

- [Gotsman:07]: Formulate in terms of angle deficit at interior/boundary vertices
  - Single linear system (almost...)

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Main References

- L. Kharevych, B. Springborn, and P. Schröder. Discrete conformal mappings via circle patterns. ACM Transactions on Graphics, 25(2):412-438, 2006.
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- http://alice.loria.fr/software/graphite/
- http://multires.caltech.edu/software/CircleParam/html/index.html

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