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Properties of Dual Pseudo-Splines

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Abstract

Dual pseudo-splines are a new family of refinable functions that generalize both the even degree B-splines and the limit functions of the dual $2n$-point subdivision schemes. They were introduced by Dyn et al. [10] as limits of subdivision schemes. In [10], simple algebraic considerations are needed to derive the approximation order of the members of this family. In this paper, we use Fourier analysis to derive further important properties such as regularity, stability, convergence, and linear independence.

1 Introduction

A function $\phi \in L_2(\mathbb{R})$ is called *refinable* if there exists a sequence $a \in \ell_2(\mathbb{Z})$ such that

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} a(k)\phi(2x-k), \qquad x \in \mathbb{R} \tag{1.1}$$

and the sequence a is then called the *refinement mask* of ϕ. Considering the *Fourier transform*

$$\hat{\phi}(\xi) = \int_{\mathbb{R}} \phi(t)e^{-i\xi t}\,dt, \qquad \xi \in \mathbb{R}$$

of ϕ and the *Fourier series*

$$\hat{a}(\xi) = \sum_{k\in\mathbb{Z}} a(k)e^{-i\xi k}, \qquad \xi \in \mathbb{R}$$

of a, the refinement equation (1.1) can be restated in the Fourier domain as

$$\hat{\phi}(\xi) = \hat{a}(\xi/2)\hat{\phi}(\xi/2). \tag{1.2}$$

Under the assumption that only finitely many elements of a are non-zero and $\hat{a}(0) \neq 0$, Eq. (1.1) has a unique (distribution) solution (up to scalar multiplication), hence a refinable function is often defined in terms of its refinement mask a or the Fourier series $\hat{a}$, which we call a refinement mask, too, for convenience. Alternatively, it may also be given in terms of its *symbol*

$$\tilde{a}(z) = \sum_{k\in\mathbb{Z}} a(k)z^k, \qquad z \in \mathbb{C} \setminus \{0\}$$

which relates to the corresponding Fourier series by

$$\hat{a}(\xi) = \tilde{a}(e^{-i\xi}), \qquad \xi \in \mathbb{R}. \tag{1.3}$$

Refinable functions have been studied intensively over the last 30 years, both in the context of subdivision and wavelets. On the one hand, any convergent stationary subdivision scheme with a finite mask a yields a compactly supported refinable function ϕ as its *basic limit function* [11], and on the other hand, refinable functions lead to the constructions of wavelets, framelets, and filter designs [2]. In both cases, amongst the most important properties of a refinable function ϕ are its smoothness and support size and the approximation order of the related subdivision scheme or wavelet system.

Recently, Dong and Shen [7] introduced the family of *pseudo-splines* (of second type) which provides a wide range of refinable functions and allows to meet various demands for balancing the approximation order, the support size and the regularity in applications. Pseudo-splines (of first type) were previously introduced by Daubechies et al. [3] to construct tight wavelet frames with high approximation order via the unitary extension principle of Ron and Shen [18]. The general refinement mask of a pseudo-spline (of second type) $\phi_{m,l}$ of order (m,l) with $0 \le l \le m-1$ is

$$\hat{a}_{m,l}(\xi) = \cos^{2m}(\xi/2) \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2) \cos^{2(l-j)}(\xi/2), \tag{1.4}$$

and this family neatly fills the gap between the odd degree B-splines and the interpolatory Dubuc–Deslauriers functions [5, 8], which turn out to be pseudo-splines of orders $(m,0)$ and $(m,m-1)$, respectively. While B-splines stand out due to their high smoothness and short support, they provide a rather poor approximation order; in contrast, the interpolatory refinable functions have optimal approximation order but low smoothness and large support. Dong and Shen [7] show that the pseudo-splines in between balance these properties as follows: for fixed m, the smoothness of the pseudo-spline of order (m,l) decreases with l, while its support and approximation order increase with l. Moreover, they prove that all pseudo-splines are stable and that their shifts are linearly independent [6], and they derive lower bounds on the regularity exponent of all pseudo-splines [7].

In the context of curve subdivision, these pseudo-splines relate to *primal schemes* in the sense that the refinement rule for each subdivision step can be split into one rule for determining the new vertices (one for each current edge) and another that describes how to compute new positions for the current vertices. On the other hand, *dual schemes* can be interpreted as generating two new vertices for each current edge and discarding all current vertices. The rules for both new vertices are usually symmetric to each other. Prominent examples of such dual schemes are the even degree B-splines and the dual $2n$-point subdivision schemes of Dyn et al. [9], which constitute the dual counterpart of the Dubuc–Deslauriers schemes. As in the primal setting, all these schemes are just special members of a whole family of dual schemes. This family of *dual pseudo-splines* was first introduced by Dyn et al. [10].

The general refinement mask of a dual pseudo-spline $\psi_{m,l}$ of order (m,l) with $0 \le l \le m-1$ is

$$\hat{b}_{m,l}(\xi) = e^{i\xi/2} \cos^{2m+1}(\xi/2) \sum_{j=0}^{l} \binom{m+1/2+l}{j} \sin^{2j}(\xi/2) \cos^{2(l-j)}(\xi/2). \tag{1.5}$$

Dyn et al. [10] observe that $\psi_{m,0}$ is the B-spline of degree $2m$ and that $\psi_{m,m-1}$ is the limit function of the dual $2m$-point scheme. It is also shown in [10], by simple algebraic arguments, that the general dual pseudo-spline $\psi_{m,l}$ provides the same approximation order as the (primal) pseudo-spline $\phi_{m,l}$, namely $2l+2$.

In this paper we use Fourier analysis and extend the results of Dong and Shen [6, 7] to dual pseudo-splines. In Section 2 we show that they are continuous and stable, and conclude the convergence of the corresponding subdivision schemes. In Section 3 we show that the integer shifts of any dual pseudo-spline are linearly independent. Finally, we derive lower bounds for the regularity exponents of the dual pseudo-splines in Section 4.

2 Stability and Convergence of the Subdivision Schemes

Before we show the stability of dual pseudo-splines we first need to verify that they belong to $L_2(\mathbb{R})$. Substituting $x = \cos^2(\xi/2)$ in Eqs. (1.4) and (1.5) we have

$$\begin{aligned} |\hat{b}_{m,l}(\xi)| &= x^{m+1/2} \sum_{j=0}^{l} \binom{m+1/2+l}{j} (1-x)^j x^{l-j} \\ &\le x^{m+1/2} \sum_{j=0}^{l} \binom{m+1+l}{j} (1-x)^j x^{l-j} \\ &= x^{-1/2} |\hat{a}_{m+1,l}(\xi)| \end{aligned} \tag{2.1}$$

which allows to derive a rough estimate for the regularity of $\psi_{m,l}$ from the regularity of $\phi_{m+1,l}$.

Corollary 2.1. *All dual pseudo-splines are compactly supported continuous functions in $L_2(\mathbb{R})$.*

Proof. According to Dong and Shen [7, Theorem 3.4] the decay of the Fourier transform of the pseudo-spline $\phi_{m,l}$ can be estimated by

$$|\hat{\phi}_{m,l}(\xi)| \leq C(1+|\xi|)^{-\beta_{m,l}}$$

with

$$\beta_{m,l} = 2m + 2l - \log\big(\textstyle\sum_{j=0}^{l} \binom{m+l}{j} 3^j\big) / \log 2.$$

From (1.2) and (2.1) we conclude that the decay rate of $\hat{\psi}_{m,l}$ is bounded by

$$|\hat{\psi}_{m,l}(\xi)| \leq C(1+|\xi|)^{-\beta_{m+1,l}+1}, \tag{2.2}$$

so the regularity exponent of the dual pseudo spline $\psi_{m,l}$ is at least $\beta_{m+1,l} - 2 - \varepsilon$. Since Dong and Shen [7, Proposition 3.5] further show that all $\beta_{m+1,l}$ for $0 \leq l \leq m-1$ are not smaller than $\beta_{2,1} = 2.67807$ it follows that all dual pseudo-splines are at least continuous functions. Moreover, they are compactly supported because their refinement masks are finite and thus they belong to $L_2(\mathbb{R})$. $\square$

In Section 4 we present a more detailed regularity analysis of dual pseudo-splines by deriving the exact decay rate of their Fourier transforms, but the above rough estimate suffices to conclude the stability of all dual pseudo-splines and the convergence of the corresponding subdivision schemes. First we recall that a function $\phi \in L_2(\mathbb{R})$ is called stable if there exist two positive constants A_1, B_1, such that for any sequence $c \in \ell_2(\mathbb{Z})$

$$A_1 \|c\|_{\ell_2(\mathbb{Z})} \leq \Big\| \sum_{i\in\mathbb{Z}} c(i)\phi(\cdot - i) \Big\|_{L_2(\mathbb{R})} \leq B_1 \|c\|_{\ell_2(\mathbb{Z})}.$$

It is well known [2, 4, 13, 17] that this condition is equivalent to the existence of two other positive constants A_2, B_2, such that

$$A_2 \leq \sum_{k\in\mathbb{Z}} \big|\hat{\phi}(\xi + 2\pi k)\big|^2 \leq B_2 \tag{2.3}$$

for almost all $\xi \in \mathbb{R}$. The upper bound always holds if ϕ has compact support and the lower bound is equivalent to

$$\big(\hat{\phi}(\xi + 2\pi k)\big)_{k\in\mathbb{Z}} \neq \mathbf{0}, \qquad \text{for all } \xi \in \mathbb{R}, \tag{2.4}$$

where $\mathbf{0}$ denotes the zero sequence.

Lemma 2.2. *All dual pseudo-splines are stable.*

Proof. Since dual pseudo-splines are compactly supported and belong to $L_2(\mathbb{R})$, as shown in Corollary 2.1, we only need to show that (2.4) holds. By (1.5) it is clear that the refinement mask of the dual pseudo-spline $\psi_{m,l}$ satisfies

$$|\hat{b}_{m,l}(\xi)| \geq |\cos^{2m+1}(\xi/2)|$$

for all $\xi \in \mathbb{R}$. And since

$$\hat{\psi}_{m,l}(\xi) = \prod_{j=1}^{\infty} \hat{b}_{m,l}(2^{-j}\xi),$$

it follows that

$$|\hat{\psi}_{m,l}(\xi)| \geq |\hat{B}_{2m+1}(\xi)|$$

for all $\xi \in \mathbb{R}$, where B_{2m+1} denotes the B-spline of order $2m+1$. But as B-splines are stable and so (2.4) holds for $\phi = B_{2m+1}$ we conclude that this condition also holds for $\phi = \psi_{m,l}$. $\square$

Now, with the stability of dual pseudo-splines and the fact that they are continuous, we conclude the following statement (see [1] for a proof).

Corollary 2.3. *The subdivision schemes that correspond to the masks of the dual pseudo-splines are convergent.*

3 Linear Independence

Let us now turn to show the linear independence of the shifts of dual pseudo-splines. A compactly supported function $\phi \in L_2(\mathbb{R})$ and its shifts are *linearly independent* if and only if

$$\sum_{j\in\mathbb{Z}} c(j)\phi(\cdot - j) = 0 \qquad \Longrightarrow \qquad c = \mathbf{0}$$

for all sequences $c \in \ell_2(\mathbb{Z})$. Based on the results of Jia and Wang [14], Dong and Shen [6, Lemma 2.1] show that in case ϕ is stable, the linear independence of its shifts is equivalent to the property that the corresponding symbol $\tilde{a}$ does not have any symmetric zeros in $\mathbb{C} \setminus \{0\}$. Due to Lemma 2.2 it remains to verify that this condition holds for the symbols $\tilde{b}_{m,l}$ of the dual pseudo-splines in order to prove the following theorem.

Theorem 3.1. *The shifts of any dual pseudo-spline are linearly independent.*

Proof. Substituting $z = y^2$ and $y = e^{-i\xi/2}$ in Eq. (1.5) and recalling relation (1.3), the symbol $\tilde{b}_{m,l}$ of the dual pseudo-spline of order (m, l) can be written as

$$\begin{aligned}
\tilde{b}_{m,l}(z) &= y^{-1}\left(\frac{1+y^2}{2y}\right)^{2m+1}\sum_{j=0}^{l}\binom{m+1/2+l}{j}\left(\frac{1-y^2}{2i\,y}\right)^{2j}\left(\frac{1+y^2}{2y}\right)^{2(l-j)} \\
&= y^{-1}\left(\frac{1+y^2}{2y}\right)^{2m+2l+1}\sum_{j=0}^{l}\binom{m+1/2+l}{j}\left(\frac{1-y^2}{i(1+y^2)}\right)^{2j} \\
&= \frac{(1+z)^{2m+2l+1}}{2z(4z)^{m+l}}\sum_{j=0}^{l}\binom{m+1/2+l}{j}\left(-\frac{(1-z)^2}{(1+z)^2}\right)^{j}.
\end{aligned}$$

Obviously, $z = -1$ is a zero of $\tilde{b}_{m,l}(z)$, but it is not a symmetric zero, because $\tilde{b}_{m,l}(1) = 1$. Hence it suffices to show that the polynomial

$$q(z) = \sum_{j=0}^{l}\binom{m+1/2+l}{j}\left(-\frac{(1-z)^2}{(1+z)^2}\right)^{j}$$

has no symmetric zeros in $\mathbb{C}\setminus\{0, 1, -1\}$. But as the coefficients $c_j = \binom{m+1/2+l}{j}$ of q form a strictly positive and increasing sequence

$$0 < c_0 < c_1 < \cdots < c_l \tag{3.1}$$

for any m, l with $0 \le l \le m - 1$, we can utilize Proposition 2.3 of [6] which states that the zeros of any polynomial $p(y) = \sum_{j=0}^{l} c_j y^j$ with coefficients satisfying (3.1) are contained in the open unit disk $D = \{y \in \mathbb{C} : |y| < 1\}$. Now suppose z_0 and $-z_0$ are symmetric zeros of q, then

$$y_0 = -\frac{(1-z_0)^2}{(1+z_0)^2} \qquad \text{and} \qquad y_1 = -\frac{(1+z_0)^2}{(1-z_0)^2}$$

must both be contained in D, and therefore

$$|y_0|\,|y_1| < 1,$$

but this contradicts the fact that

$$|y_0|\,|y_1| = 1.$$

It follows that $q(z)$ does not have any symmetric zeros in $\mathbb{C} \setminus \{0, 1, -1\}$ and that $\tilde{b}_{m,l}(z)$ does not have any symmetric zeros in $\mathbb{C} \setminus \{0\}$, which concludes the proof. $\square$

4 Regularity

In this section we derive lower bounds for the regularity exponents $\alpha_{m,l}$ of the dual pseudo-splines $\psi_{m,l}$ by estimating the decay rates $\gamma_{m,l}$ of their Fourier transforms and using the relation that

$$\alpha_{m,l} \geq \gamma_{m,l} - 1 - \varepsilon,$$

for any positive ε small enough [2]. In the following we refer to the numbers $\gamma_{m,l} - 1$ as the *lower bounds of regularity* of the dual pseudo-splines $\psi_{m,l}$.

It is well known [2, 15] that the refinement mask a of any compactly supported refinable function $\phi \in L_2(\mathbb{R})$ with $\hat{\phi}(0) = 1$ must satisfy $\hat{a}(0) = 1$ and $\hat{a}(\pi) = 0$, so that $\hat{a}(\xi)$ can be factorized as

$$\hat{a}(\xi) = \cos^n(\xi/2)\mathcal{L}(\xi),$$

where n is the maximal multiplicity of the zero of $\hat{a}$ at π, and $\mathcal{L}(\xi)$ is a trigonometric polynomial satisfying $\mathcal{L}(0) = 1$. Hence, we have

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} \hat{a}(2^{-j}\xi) = \prod_{j=1}^{\infty} \cos^n(2^{-j}\xi/2) \prod_{j=1}^{\infty} \mathcal{L}(2^{-j}\xi) = \operatorname{sinc}^n(\xi/2) \prod_{j=1}^{\infty} \mathcal{L}(2^{-j}\xi),$$

which suggests that the decay of $|\hat{\phi}|$ can be characterized by $|\mathcal{L}(\xi)|$ as stated in the following lemma, taken from Daubechies [2, Lemma 7.1.7].

Lemma 4.1. *Let ϕ be a refinable function with a refinement mask a of the form*

$$|\hat{a}(\xi)| = \cos^n(\xi/2)|\mathcal{L}(\xi)|, \qquad \xi \in [-\pi, \pi], \quad n \in \mathbb{N} \tag{4.1}$$

and suppose that the two inequalities

$$\begin{aligned} |\mathcal{L}(\xi)| &\leq |\mathcal{L}(\tfrac{2\pi}{3})|, && \textit{for } |\xi| \in [0, \tfrac{2\pi}{3}], \\ |\mathcal{L}(\xi)\mathcal{L}(2\xi)| &\leq |\mathcal{L}(\tfrac{2\pi}{3})|^2, && \textit{for } |\xi| \in [\tfrac{2\pi}{3}, \pi] \end{aligned} \tag{4.2}$$

hold. Then $|\hat{\phi}(\xi)| \leq C(1 + |\xi|)^{-n+\kappa}$ with $\kappa = \log(|\mathcal{L}(\frac{2\pi}{3})|)/\log 2$, and this decay is optimal.

Using this lemma and closely following the approach of Dong and Shen [7], it is straightforward to derive the optimal decay rate of the Fourier transforms of dual pseudo-splines. For that we use the observation of Dyn et al. [10], that the refinement mask of the dual pseudo spline $\psi_{m,l}$ in (1.5) can also be written as

$$\hat{b}_{m,l}(\xi) = e^{i\xi/2} \cos^{2m+1}(\xi/2) \sum_{j=0}^{l} \binom{m - 1/2 + j}{j} \sin^{2j}(\xi/2). \tag{4.3}$$

Letting $r = m + 1/2$ and $y = \sin^2(\xi/2)$ we then have

$$|\hat{b}_{m,l}(y)| = (1 - y)^r P_{r,l}(y)$$

with

$$P_{r,l}(y) = \sum_{j=0}^{l} \binom{r - 1 + j}{j} y^j, \tag{4.4}$$

which verifies that the refinement mask $b_{m,l}$ of $\psi_{m,l}$ is of the form (4.1) with $|\mathcal{L}(\xi)| = P_{r,l}(y)$ and $n = 2m+1$.

Proposition 4.2. *Let $P_{r,l}(y)$ be defined as in (4.4) with $r \in \mathbb{R}$ and $0 \leq l \leq \lfloor r \rfloor - 1$. Then*

$$\begin{aligned} P_{r,l}(y) &\leq P_{r,l}(\tfrac{3}{4}) && \textit{for } y \in [0, \tfrac{3}{4}] \\ P_{r,l}(y)P_{r,l}(4y(1-y)) &\leq \left(P_{r,l}(\tfrac{3}{4})\right)^2 && \textit{for } y \in [\tfrac{3}{4}, 1]. \end{aligned}$$

(m,l)	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$
$m=1$	2							
$m=2$	4	2.4764						
$m=3$	6	4.1420	2.9902					
$m=4$	8	5.8707	4.4972	3.4987				
$m=5$	10	7.6424	6.0759	4.9112	3.9982			
$m=6$	12	9.4454	7.7083	6.3935	5.3472	4.4894		
$m=7$	14	11.2721	9.3821	7.9309	6.7619	5.7926	4.9733	
$m=8$	16	13.1174	11.0890	9.5129	8.2302	7.1568	6.2418	5.4513

Table 1: Lower bounds of regularity of the dual pseudo-splines $\psi_{m,l}$ as derived from the decay rates of their Fourier transforms.

Proof. The first inequality clearly holds because $P_{r,l}(y)$ is monotonically increasing for $y \geq 0$ and the second inequality can be proven by following exactly the same arguments as in the proof of Proposition 3.2 in [7]. $\square$

Taking $r = m + 1/2$ in Proposition 4.2 and applying Lemma 4.1, we now obtain an estimate for the regularity exponent of all dual pseudo-splines.

Theorem 4.3. *Let $\psi_{m,l}$ be the dual pseudo-spline of order (m,l) with $0 \leq l \leq m-1$. Then*

$$|\hat{\psi}_{m,l}| \leq C(1+|\xi|)^{-\gamma_{m,l}}$$

with $\gamma_{m,l} = 2m + 1 - \log\big(P_{r,l}(\frac{3}{4})\big)/\log 2$, and the decay rate $\gamma_{m,l}$ is optimal. Consequently, $\psi_{m,l} \in C^{\alpha_{m,l}}$ with $\alpha_{m,l} \geq \gamma_{m,l} - 1 - \varepsilon$ for any $\varepsilon > 0$.

Table 1 lists the lower bounds of regularity $\gamma_{m,l} - 1$ of the dual pseudo-splines $\psi_{m,l}$ with $1 \leq m \leq 8$ and $0 \leq l \leq m-1$. The table shows that this lower bound decreases as l increases for fixed m, while for fixed l it increases with m. This observation is verified by the following proposition.

Proposition 4.4. *Let $\gamma_{m,l}$ be the decay rate of the Fourier transform of the dual pseudo-spline $\psi_{m,l}$ as given in Theorem 4.3. Then the following statements hold:*

1. *For fixed m, $\gamma_{m,l}$ decreases as l increases.*
2. *For fixed l, $\gamma_{m,l}$ increases as m increases.*
3. *When $l = m-1$, $\gamma_{m,l}$ increases as m increases.*

Consequently, the decay rate $\gamma_{2,1} = 3.4764$ is the smallest among all $\gamma_{m,l}$ with $m \geq 2$ and $0 \leq l \leq m-1$.

Proof. The proof is analogous to that of Proposition 3.5 in [7]. $\square$

From Theorem 4.3 we also conclude the asymptotic behaviour of the decay rate $\gamma_{m,l}$.

Corollary 4.5. *Let $\psi_{m,l}$ be the dual pseudo-spline of order (m,l) and consider the case where $l = \lfloor \lambda m \rfloor$ for $\lambda \in [0,1]$. Then*

$$|\hat{\psi}_{m,l}(\xi)| \leq C(1+|\xi|)^{-\mu m}, \qquad \text{with } \mu = \log(\tfrac{4}{1+\lambda})^{\lambda+1}(\tfrac{\lambda}{3})^{\lambda}/\log 2,$$

asymptotically for large m.

Proof. Since $\log\binom{m+1/2+l}{l} \sim \log\binom{m+l}{l}$ asymptotically for large m and l, the proof follows from that of Theorem 3.6 in [7] by replacing m with $m + 1/2$. $\square$

We call the value μ from Corollary 4.5 the *asymptotic decay rate* for dual pseudo-splines and note that it is exactly the same as the one for pseudo-splines (see Theorem 3.6 in [7]). However, as we can see from Table 1, the dual pseudo-spline of order (m,l) is smoother than the pseudo-spline of order (m,l) for small m and l (see Table 1 in [7]), yet its support is larger by one. This observation is confirmed

(m,l)	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$
$m=1$	2							
$m=2$	4	2.8301						
$m=3$	6	4.5406	3.5772					
$m=4$	8	6.2996	5.1271	4.2473				
$m=5$	10	8.0931	6.7358	5.6935	4.8542			
$m=6$	12	9.9125	8.3899	7.1998	6.2268	5.4114		
$m=7$	14	11.7521	10.0804	8.7549	7.6581	6.7293	5.9328	
$m=8$	16	13.6077	11.8003	10.3503	9.1386	8.1038	7.2097	6.4307

Table 2: Regularity exponents of the dual pseudo-splines $\psi_{m,l}$ as derived from joint spectral radius analysis. While the values for $l=0$ and $l=1$ are exact, the others are derived numerically and are just lower bounds of regularity; cf. Table 1.

(m,l)	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$
$m=1$	1							
$m=2$	3	2						
$m=3$	5	3.6781	2.8301					
$m=4$	7	5.4150	4.3438	3.5511				
$m=5$	9	7.1926	5.9250	4.9621	4.1936			
$m=6$	11	9	7.5578	6.4400	5.5325	4.7767		
$m=7$	13	10.8301	9.2311	7.9719	6.9358	6.0627	5.3173	
$m=8$	15	12.6781	10.9370	9.5480	8.3927	7.4101	6.5640	5.8294

Table 3: Regularity exponents of the pseudo-splines $\phi_{m,l}$ as derived from joint spectral radius analysis. While the values for $l=0$ and $l=1$ are exact, the others are derived numerically and are just lower bounds of regularity; cf. Table 1 in [7].

by the values given in Tables 2 and 3. Here we used joint spectral radius analysis [11, 16] to derive the regularity exponents. While the values for the B-splines ($l=0$) and the pseudo-splines with cubic precision ($l=1$) are exact (see Theorem 1 in [12]), the others are lower bounds that we computed numerically by approximating the joint spectral radius. Note that the joint spectral radius analysis yields lower bounds that are better then those concluded from the decay rate of the Fourier transform.

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