Multinode Rational Operators for Univariate Interpolation

Francesco Dell’Accio\textsuperscript{1,}\textsuperscript{b)}, Filomena Di Tommaso\textsuperscript{1,}\textsuperscript{a)} and Kai Hormann\textsuperscript{2,}\textsuperscript{c)}

\textsuperscript{1}Dipartimento di Matematica e Informatica, Università della Calabria, Italy
\textsuperscript{2}Faculty of Informatics, Università della Svizzera italiana, Lugano, Switzerland
\textsuperscript{a)Corresponding author: ditommaso@mat.unical.it}
\textsuperscript{b)fdellacci@unical.it}
\textsuperscript{c)kai.hormann@usi.ch}

Abstract. Birkhoff (or lacunary) interpolation is an extension of polynomial interpolation that appears when observation gives irregular information about function and its derivatives. A Birkhoff interpolation problem is not always solvable even in the appropriate polynomial or rational space. In this talk we split up the initial problem in subproblems having a unique polynomial solution and use multinode rational basis functions in order to obtain a global interpolant.

INTRODUCTION

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a set of pairwise distinct real numbers for which we assume that \( x_1 < x_2 < \ldots < x_n \). In the problem of interpolation of given data \( f_{i,j} = f^{(j)}(x_i), i = 1, \ldots, n, j \in \mathcal{J}_i \subset \mathbb{N} \), by a polynomial \( p \) of appropriate degree,

\[
p^{(j)}(x_i) = f_{i,j}
\]

we mainly distinguish between Hermite interpolation and Birkhoff interpolation. We have an Hermite interpolation problem if, for each \( i \), the indices \( j \) in the set \( \mathcal{J}_i \) form an unbroken sequence, i.e. \( \mathcal{J}_i = \{0, 1, \ldots, \mu_i\} \), Birkhoff interpolation otherwise. It is, however, convenient to consider Hermite interpolation to be a special case of lacunary interpolation and to deal with Hermite-Birkhoff interpolation. In contrast to Hermite interpolation, a Birkhoff interpolation problem does not always have a unique solution or, even worse, does not have a solution \([1]\). In this paper we propose to split up the unsolvable problems in two or more uniquely solvable subproblems, whose solutions can be blended together. Here we consider the case of multinode basis functions \([2]\) as blending functions. An approach to Birkhoff interpolation using Shepard basis functions can be found in \([3, 4, 5, 6, 7, 8, 9]\). To this goal we consider a covering \( \mathcal{F} = \{F_1, F_2, \ldots, F_m\} \) of \( X \) by subsets \( F_k \subset X \) such that, for each \( k = 1, \ldots, m \), the corresponding Hermite-Birkhoff interpolation subproblems \( p^{(j)}(x_i) = f_{i,j}, x_i \in F_k, j \in \mathcal{J}_i \) have a unique solution and we associate to each \( F_k, k = 1, \ldots, m \), a multinode basis function. The latter are then used in combination with the local Hermite-Birkhoff polynomials that interpolate the data associated to \( F_k \). Finally, we provide numerical experiments which show the approximation order.

MULTINOSE BASIS FUNCTIONS

Let us consider a covering \( \mathcal{F} = \{F_1, F_2, \ldots, F_m\} \) of \( X \) by its not empty subsets \( F_k \subset X \), that is

\[
\bigcup_{k=1}^{m} F_k = X, \quad F_k \neq \emptyset, \quad \text{for each } k = 1, \ldots, m. \tag{1}
\]

The multinode basis functions with respect to the covering \( \mathcal{F} \) are defined by

\[
B_{\mu,k}(x) = \frac{\prod_{i \in \mathcal{J}_k, i \neq \mu} |x-x_i|^{\mu - i}}{\sum_{i \in \mathcal{J}_k, i \neq \mu} |x-x_i|^{\mu - i}}, \quad k = 1, \ldots, m. \tag{2}
\]
where $\mu > 0$ is a parameter that determines the differentiability class of the basis and controls the range of influence of the data values. The multinode basis functions (2) are non-negative and form a partition of unity, that is
\[ \sum_{k=1}^{m} B_{\mu,k}(x) = 1; \]
but instead of being cardinal they vanish at all nodes $x_j$ that are not in $F_k$, that is
\[ B_{\mu,k}(x_j) = 0, \quad \mu > 0, \]
for any $k = 1, \ldots, m$ and $j \not\in F_k$, and
\[ \sum_{k \in K_i} B_{\mu,k}(x_i) = 1, \quad \mu > 0 \]
where
\[ K_i = \{ l \in \{1, \ldots, m\} : x_l \in F_i \} \neq \emptyset, \]
is the set of indices of all subsets of $\mathcal{F}$ that contain $x_i$. For $\mu > 0$ even integer the multinode basis functions (2) are rational and have no real poles, otherwise their class of differentiability is $\mu - 1$ for $\mu$ odd integer and $[\mu]$, the largest integer not greater than $\mu$, in all remaining cases. Moreover, all derivatives of order $\ell > 0$ vanish at all nodes $x_j$ that are not in $F_k$,
\[ B^{(\ell)}_{\mu,k}(x_j) = 0, \quad \mu > 0, \]
for any $k = 1, \ldots, m$ and $j \not\in F_k$ and
\[ \sum_{k \in K_i} B^{(\ell)}_{\mu,k}(x_i) = 0, \quad \mu > 1. \]

**MULTINOIDE GLOBAL INTERPOLATION OPERATOR**

Let us consider the Hermite-Birkhoff interpolation problem
\[ p^{(j)}[f](x_i) = f^{(j)}(x_i), \quad i = 1, \ldots, n, \quad j \in \mathcal{J}_i, \]
and let us assume that, for each $k = 1, \ldots, m$, the Hermite-Birkhoff interpolation subproblems
\[ P_k^{(j)}[f](x_i) = f^{(j)}(x_i), \quad x_i \in F_k, \quad j \in \mathcal{J}_i, \]
have a unique solution $P_k[f]$ in their appropriate polynomial spaces $\mathcal{P}^\mu_{\mathcal{F}_k}$, $q_k = \sum_{x_i \in F_k} \#(\mathcal{J}_i) - 1$. As soon as we have provided a solution for all local Hermite-Birkhoff interpolation problems, we define the multinode global interpolation operator by
\[ M_\mu[f, \mathcal{F}](x) = \sum_{k=1}^{m} B_{\mu,k}(x) P_k[f](x) \]
where $P_k[f](x)$ is the polynomial solution of the Hermite-Birkhoff interpolation problem on $F_k$. The operator $M_\mu[f, \mathcal{F}](x)$ has remarkable properties. Firstly, it reproduces polynomials up to the degree $q_{\min} = \min_k q_k$ and by setting $\mathcal{F} = \{x\}$, $M_\mu[f, \mathcal{F}](x)$ coincides with that polynomial solution if the global problem has a unique polynomial solution. Secondly, the operator $M_\mu[f, \mathcal{F}]$ interpolates the functional data
\[ M_\mu[f, \mathcal{F}](x_i) = f(x_i), \quad \text{for each} \ i : 0 \in \mathcal{J}_i \]
and, if $\mathcal{F}$ is a partition of $X$ (i.e. $F_\alpha \cap F_\beta = \emptyset$ for each $\alpha \neq \beta$) the operator $M_\mu[f, \mathcal{F}]$ interpolates all data used in its definition, i.e.
\[ M_\mu^{(j)}[f, \mathcal{F}](x_i) = f^{(j)}(x_i), \quad \text{for each} \ k = 1, \ldots, m, \ x_i \in F_k, \ j \in \mathcal{J}_i. \]
However, we notice that the operator $M_\mu[f, \mathcal{F}]$ could not interpolate all derivative data at some $x_k$ if $\sharp(K_k) > 1$ and the sequence of indices in $\mathcal{J}_k$ is broken. For example, let us assume
\[ \sharp(K_k) = 2, \quad F_\alpha \cap F_\beta = \{x_k\}, \quad \mathcal{J}_k = \{0, 2, \ldots, \ell - 1, \ell\}, \ell \geq 2 \]
and
\[ B^{(l-1)}_{\mu,\alpha}(x_k)P'_{\alpha}[f](x_k) + B^{(l-1)}_{\mu,\beta}(x_k)P'_{\beta}[f](x_k) \neq 0. \]

We notice that
\[ P'_{\alpha}[f](x_k) \neq P'_{\beta}[f](x_k) \]

since property (8). From
\[ P^{(l)}_{\alpha}[f](x_k) = P^{(l)}_{\beta}[f](x_k) = f^{(l)}(x_k) \]

by properties (4) and (5) easily follows
\[ \sum_{k=1}^{m} B_{\mu,\alpha}(x_k)P^{(l)}_{\alpha}[f](x_k) = B_{\mu,\alpha}(x_k)f^{(l)}(x_k) + B_{\mu,\beta}(x_k)f^{(l)}(x_k) = f^{(l)}(x_k). \]

On the other hand
\[ \sum_{k=1}^{m} \sum_{i=0}^{l-1} \left( \sum_{i=0}^{l-1} \left( B^{(l-1)}_{\mu,\alpha}(x_k)P^{(l)}_{\alpha}[f](x_k) + B^{(l-1)}_{\mu,\beta}(x_k)P^{(l)}_{\beta}[f](x_k) \right) \right) \]

by property (7). Let us fix our attention to the right hand side of previous equality. For each \( i \in J \), we get
\[ B^{(l-1)}_{\mu,\alpha}(x_k)P^{(l)}_{\alpha}[f](x_k) + B^{(l-1)}_{\mu,\beta}(x_k)P^{(l)}_{\beta}[f](x_k) = \left( B^{(l-1)}_{\mu,\alpha}(x_k) + B^{(l-1)}_{\mu,\beta}(x_k) \right) f^{(l)}(x_k) = 0 \]

by property (8), but
\[ B^{(l-1)}_{\mu,\alpha}(x_k)P^{(l)}_{\alpha}[f](x_k) + B^{(l-1)}_{\mu,\beta}(x_k)P^{(l)}_{\beta}[f](x_k) \neq 0 \]

and consequently
\[ M^{(l)}_{\mu}[f,\mathcal{F}](x_k) \neq f^{(l)}(x_k). \]

In order to avoid this trouble, we proceed as follows. For each \( \kappa = 1, \ldots, n \) let be \( v_\kappa = \#(K_\kappa) \) and \( F_{\alpha_1}, \ldots, F_{\alpha_n} \) the subset of \( X \) which contain \( x_k \). As above, let us denote by \( P_{\alpha_1}[f], \ldots, P_{\alpha_n}[f] \) the polynomial solutions of the Hermite-Birkhoff interpolation problems on \( F_{\alpha_1}, \ldots, F_{\alpha_n} \) respectively. For all \( j = 0, 1, \ldots, \max(J) \) we set
\[ \tilde{f}^{(j)}(x_k) = \frac{1}{v_\kappa} \left( P^{(j)}_{\alpha_1}[f](x_k) + \ldots + P^{(j)}_{\alpha_n}[f](x_k) \right) \]

and we note that
\[ \tilde{f}^{(j)}(x_k) = f^{(j)}(x_k) \]

as soon as \( j \in J \). For each \( k = 1, \ldots, m \) we call the Hermite interpolation problem
\[ \tilde{f}^{(j)}_k[f](x_k) = \tilde{f}^{(j)}(x_k), \quad x_k \in F_k, \ j = 0, 1, \ldots, \max(J), \]

**hermitian completion** of the Hermite-Birkhoff interpolation problem (10). It is well known that each interpolation problem (15) has a unique solution \( \tilde{P}_k[f](x_k) \) in the polynomial space \( \mathcal{P}_k, \), \( d_k = \#(F_k) + \sum_{x \in F_k} \max(J) - 1 \), for which there are explicit formulas in Lagrange or Newton form [10]. Nevertheless, if \( q_k < d_k \) and \( p \in \mathcal{P}_k, \tilde{P}_k[p] \) may be different from \( p \), since we have completed the lacunary data using solutions of several interpolation problems. We set
\[ \tilde{M}_\mu[f,\mathcal{F}](x_k) = \sum_{k=1}^{m} B_{\mu,\alpha}(x_k)\tilde{P}_k[f](x_k). \]

The operator \( \tilde{M}_\mu[\cdot,\mathcal{F}] \) preserves the reproducing polynomial property of \( M_\mu[\cdot,\mathcal{F}] \), that is reproduces polynomials up to the degree \( q_{\min} = \min_k q_k \) and interpolates all data used in its definition, that is
\[ \tilde{M}_\mu^{(j)}[f,\mathcal{F}](x_k) = f^{(j)}(x_k), \quad \text{for each} \ k = 1, \ldots, m, \ x_k \in F_k, \ j \in J. \]
NUMERICAL RESULTS

In the following we numerically test the approximation order of the multinode rational interpolation operator. We carried out a series of experiments with different sets of equispaced nodes on [0, 1] and test functions $f_i$, $i = 1, \ldots, 6$ as in [11]. More precisely, we consider different coverings $\mathcal{F}$ of the nodeset $X$ with increasing number of subsets $F_k$. For each of the 6 test functions $f_i$ we constructed the multinode rational interpolant $M_{\mu}[f_i, \mathcal{F}](x)$ and we determined the maximum approximation error $e_{\text{max}}$ by evaluating $\left| f_i(x) - M_{\mu}[f_i, \mathcal{F}](x) \right|$ at 100,000 random points $x \in [0, 1]$ and recording the maximum value. In the Figure 1 we display the log-log-plot of the approximation error $e_{\text{max}}$ over the interval width for the 6 test functions. As reference, the dotted line indicates a perfect quadratic trend.

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