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On the Lebesgue constant of barycentric rational interpolation at equidistant nodes

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Abstract

Recent results reveal that the family of barycentric rational interpolants introduced by Floater and Hormann is very well-suited for the approximation of functions as well as their derivatives, integrals, and primitives. Especially in the case of equidistant interpolation nodes, these infinitely smooth interpolants offer a much better choice than their polynomial analogue. A natural and important question regards the condition of this rational approximation method. In this paper we extend a recent study of the Lebesgue function and constant associated to Berrut's rational interpolant at equidistant nodes to the family of Floater–Hormann interpolants, which includes the former as a special case.

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1 Introduction

The approximation problem we consider is the following: suppose we want to interpolate a function $f: [a, b] \rightarrow \mathbb{R}$ by some function g from a finite-dimensional linear subspace of the normed space of $C[a, b]$ functions with maximum norm such that g interpolates f at the $n + 1$ distinct interpolation nodes $a = x_0 < x_1 < \dots < x_n = b$,

$$g(x_k) = f(x_k), \quad k = 0, \dots, n.$$

With a given set of *basis functions* b_j satisfying the *Lagrange property*,

$$b_j(x_k) = \delta_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases}$$

we may define the class of interpolants that we focus on, namely that of linear interpolants

$$g(x) = \sum_{j=0}^n b_j(x) f(x_j).$$

We stress that by linearity we mean the dependence on the data $f(x_0), \dots, f(x_n)$. Examples include (among many others) polynomial interpolation in Lagrangian form and, in particular, barycentric rational interpolation. Besides the convergence theory of g , the study of the condition of this numerical approximation method imposes itself.

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Let every data point $f(x_j)$ be given with an absolute error or perturbation of at most ε , for example due to rounding, noise, or measurement imprecision. Then the maximum distance in $[a, b]$ between the interpolant \tilde{g} of the perturbed data and the interpolant g of the exact data is bounded as

$$\max_{a \leq x \leq b} |\tilde{g}(x) - g(x)| \leq \varepsilon \max_{a \leq x \leq b} \Lambda_n(x),$$

where the function

$$\Lambda_n(x) = \sum_{j=0}^n |b_j(x)|$$

is called the *Lebesgue function* associated to g , and its maximum over the interval $[a, b]$,

$$\Lambda_n = \max_{a \leq x \leq b} \Lambda_n(x),$$

is the *Lebesgue constant* [13]. Thus, the quantity Λ_n is the worst possible error amplification and, since g is linear in the data, it coincides with the condition number of the interpolation process [12]. Throughout this paper we make use of this interpretation of the Lebesgue constant. Its original definition as the norm of the approximation operator [13] is not needed here, since we are not looking for the *best* approximation of f in some linear space.

Numerous authors derived results about the Lebesgue function and constant associated to polynomial interpolation at different kinds of nodes; see [5, 6, 17] and the references therein. It is well known [5] that the Lebesgue constant associated to polynomial interpolation at nodes distributed in any way always increases at least logarithmically with the number of nodes. Such a rate is achieved, for instance, for various kinds of Chebyshev nodes [14, 17].

In contrast to this favourable behaviour, polynomial interpolation at *equidistant nodes* has an asymptotic exponential growth rate

$$\Lambda_n \sim \frac{2^n}{en \ln(n)}$$

as $n \rightarrow \infty$. More detailed results and other approaches to describing the error amplification may be found in [9, 16, 18] and the references therein. The bad condition, together with Runge's phenomenon [7, 15], makes polynomial interpolation at equidistant nodes often useless for $n \geq 50$. In fact, interpolation at these nodes is a challenging problem, as it was shown in [12] that it is not possible to develop an interpolation method which is both well-conditioned and converges exponentially fast as the number of nodes increases. One way to overcome this restriction and to get better results is by using rational instead of polynomial interpolation at equidistant nodes.

Berrut and Mittelmann [3] determine rational interpolants with small Lebesgue constants for given nodes by numerically optimizing the denominator of the interpolant. Here we shall concentrate on the family of barycentric rational interpolants introduced by Floater and Hormann [8] with basis functions

$$b_j(x) = \frac{(-1)^j \beta_j}{x - x_j} \bigg/ \sum_{i=0}^n \frac{(-1)^i \beta_i}{x - x_i}, \quad j = 0, \dots, n. \quad (1)$$

Explicit formulas for the positive weights β_0, \dots, β_n are given in the same paper. The original construction reveals that the so-obtained rational interpolant is a blend of local polynomial interpolants of degree at most d to $d + 1$ consecutive values of the given function, and that the approximation order is $O(h^{d+1})$ as long as the interpolated function is $d + 2$ times continuously differentiable. This family of barycentric rational interpolants is well-suited for the approximation of sufficiently smooth functions [8] as well as for applications such as the approximation of derivatives, integrals, and primitives [2, 10, 11].

For $n \geq 2d$ equidistant nodes, which is the setting that we assume from now on, the weights in (1) turn out to be

$$\beta_j = \sum_{k=d}^n \binom{d}{k-j} = \begin{cases} \sum_{k=0}^j \binom{d}{k}, & \text{if } j \leq d, \\ 2^d, & \text{if } d \leq j \leq n - d, \\ \beta_{n-j}, & \text{if } j \geq n - d. \end{cases} \quad (2)$$

If $d = 0$, then all weights are equal to one, and the favourable properties of the corresponding rational interpolant were discovered numerically by Berrut [1]. Bos, De Marchi, and Hormann [4] later analysed the associated Lebesgue constant and show that it is bounded as

$$c_n \ln(n+1) \leq \Lambda_n \leq 2 + \ln(n), \quad (3)$$

where $c_n = 2n/(4 + n\pi)$ with $\lim_{n \rightarrow \infty} c_n = 2/\pi$.

The general case $d \geq 1$ needs a different approach since the study of the Lebesgue function

$$\Lambda_n(x) = \sum_{j=0}^n |b_j(x)| = \sum_{j=0}^n \frac{\beta_j}{|x - x_j|} \bigg/ \left| \sum_{j=0}^n \frac{(-1)^j \beta_j}{x - x_j} \right| \quad (4)$$

through the direct use of the basis functions (1) results in rather complicated expressions, whereas the original form of the rational interpolants as blends of polynomials allows for much shorter proofs.

The aim of this paper is to show that the Lebesgue constant associated to the family of Floater–Hormann interpolants with $d \geq 1$ grows logarithmically in the number of interpolation nodes if they are equidistant. This is achieved by establishing logarithmic upper and lower bounds in Sections 2 and 3, respectively.

2 Upper bound

The properties of barycentric rational interpolation depend only on the distance h between the nodes as the interval length remains constant. For simplicity and without loss of generality, we assume that the interpolation interval is $[0, 1]$, so that the nodes are equally spaced with distance $h = 1/n$,

$$x_k = kh = \frac{k}{n}, \quad k = 0, \dots, n.$$

We begin by deriving an upper bound of the Lebesgue constant associated to the family of Floater–Hormann interpolants with $d \geq 1$.

Theorem 1. *The Lebesgue constant associated to rational interpolation with the basis functions $b_j(x)$ in (1) at equidistant nodes satisfies*

$$\Lambda_n \leq 2^{d-1}(2 + \ln n).$$

Proof. If $x = x_k$ for $k = 0, \dots, n$, then $\Lambda_n(x) = 1$. Otherwise, $x_k < x < x_{k+1}$ for some k with $0 \leq k \leq n-1$ and we consider

$$\Lambda_n(x) = \frac{(x - x_k)(x_{k+1} - x) \sum_{j=0}^n \frac{\beta_j}{|x - x_j|}}{(x - x_k)(x_{k+1} - x) \left| \sum_{j=0}^n \frac{(-1)^j \beta_j}{x - x_j} \right|} =: \frac{N_k(x)}{D_k(x)}.$$

Since all the weights β_j are less than or equal to 2^d , the numerator is bounded as

$$N_k(x) \leq 2^d \left(\frac{1}{n} + \frac{1}{2n} \ln n \right),$$

following the proof of Theorem 2 in [4] for the case $d = 0$.

We now show that the denominator $D_k(x)$ is bounded from below by $1/n$, which leads to the claimed result. To see this, we recall from [8, Section 4] that

$$\sum_{j=0}^n \frac{(-1)^j \beta_j}{x - x_j} = (-1)^d d! h^d \sum_{j=0}^{n-d} \lambda_j(x), \quad (5)$$

where

$$\lambda_j(x) = \frac{(-1)^j}{(x - x_j) \cdots (x - x_{j+d})}.$$

Assuming $k \leq n - d$ for the moment, it is shown in the proof of Theorem 2 in [8] that

$$\left| \sum_{j=0}^{n-d} \lambda_j(x) \right| \geq |\lambda_k(x)| = \frac{1}{(x - x_k)(x_{k+1} - x) \cdots (x_{k+d} - x)}. \quad (6)$$

Therefore,

$$D_k(x) = d!h^d(x - x_k)(x_{k+1} - x) \left| \sum_{j=0}^{n-d} \lambda_j(x) \right| \geq \frac{d!h^d}{\prod_{j=k+2}^{k+d} (x_j - x)} \geq \frac{d!h^d}{\prod_{j=k+2}^{k+d} (x_j - x_k)} = h = \frac{1}{n},$$

where the last inequality follows from the fact that $x_j - x \leq x_j - x_k$ for $j \geq k + 1$. If $k > n - d$, then a similar reasoning leads to this lower bound for $D_k(x)$ by considering $\lambda_{k-d+1}(x)$ instead of $\lambda_k(x)$ in (6). \square

Note that the upper bound in Theorem 1 for $d = 1$ is identical to the upper bound for $d = 0$ in (3), which is consistent with our numerical observations that both cases have a similar Lebesgue constant; compare Figure 3 (top right) and Figure 2 in [4].

3 Lower bound

Let us now turn to the study of the lower bound of the Lebesgue constant associated to the family of Floater–Hormann interpolants. We first give a general result for any $d \geq 1$ and then derive an improved bound for the case $d = 1$, which turns out to be again very similar to the one for $d = 0$ in (3).

Theorem 2. *The Lebesgue constant associated to rational interpolation with the basis functions $b_j(x)$ in (1) at equidistant nodes satisfies*

$$\Lambda_n \geq \frac{1}{2^{d+2}} \binom{2d+1}{d} \ln \left(\frac{n}{d} - 1 \right).$$

Proof. From numerical experiments (see Figure 1), it appears that the Lebesgue function

$$\Lambda_n(x) = \frac{\sum_{j=0}^n \frac{\beta_j}{|x - x_j|}}{\left| \sum_{j=0}^n \frac{(-1)^j \beta_j}{x - x_j} \right|} =: \frac{N(x)}{D(x)} \quad (7)$$

obtains its maximum approximately halfway between x_0 and x_1 (and halfway between x_{n-1} and x_n because of the symmetry of $\Lambda_n(x)$ with respect to the mid-point of the interval). For this reason, we consider $x^* = (x_1 - x_0)/2 = 1/(2n)$ and derive a lower bound for $\Lambda_n(x^*)$.

We begin by investigating the numerator at x^* ,

$$N(x^*) = \sum_{j=0}^n \frac{\beta_j}{|x^* - x_j|} = \sum_{j=0}^n \frac{\beta_j}{\left| \frac{1}{2n} - \frac{j}{n} \right|} = 2n \sum_{j=0}^n \frac{\beta_j}{|2j - 1|}.$$

Omitting the first and last d terms and noticing that $\beta_j = 2^d$ for the remaining terms, we obtain

$$N(x^*) \geq 2n2^d \sum_{j=d}^{n-d} \frac{1}{2j-1} \geq 2n2^d \int_{d-1}^{n-d} \frac{1}{2x+1} = 2n2^d \frac{1}{2} \ln \left(\frac{2n-2d+1}{2d-1} \right) \geq n2^d \ln \left(\frac{n}{d} - 1 \right).$$

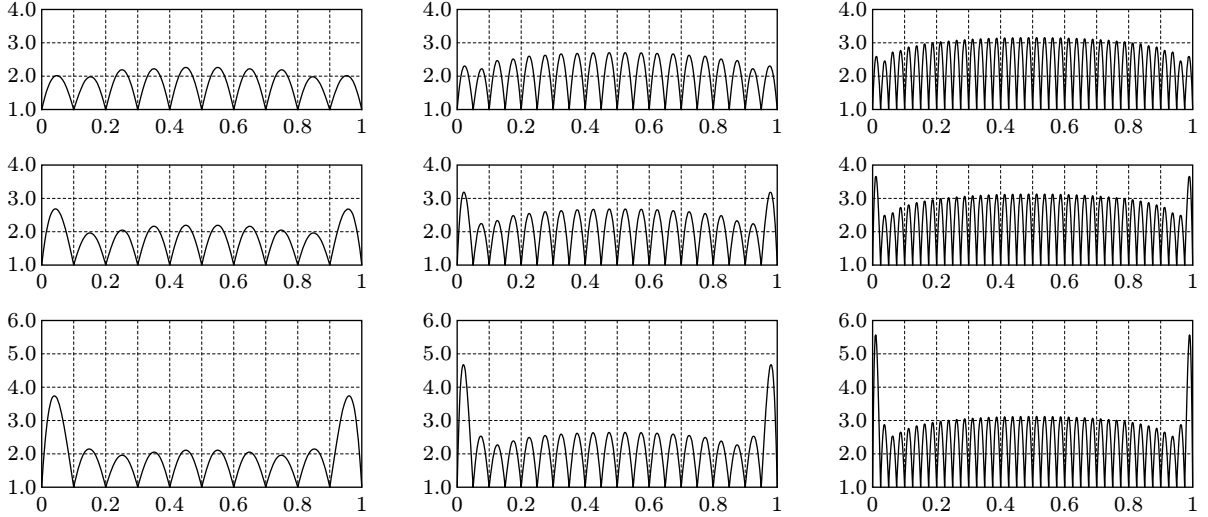


Figure 1: Lebesgue function of the Floater–Hormann interpolants for $d = 1$ (top), $d = 2$ (middle), and $d = 3$ (bottom) at $n + 1$ equidistant nodes for $n = 10, 20, 40$.

To handle the denominator, we first recall (5) to get

$$D(x^*) = \left| \sum_{j=0}^n \frac{(-1)^j \beta_j}{x^* - x_j} \right| = d! h^d \left| \sum_{j=0}^{n-d} \lambda_j(x^*) \right|.$$

As x^* belongs to the first sub-interval $[x_0, x_1]$, we deduce that $\lambda_0(x^*)$ and $\lambda_1(x^*)$ have the same sign and the following $\lambda_j(x^*)$ oscillate in sign and decrease in absolute value. The absolute value of the sum over these functions is thus bounded from above by the sum of the absolute values of the first two terms,

$$D(x^*) \leq d! h^d (|\lambda_0(x^*)| + |\lambda_1(x^*)|),$$

and expanding this expression at x^* finally gives

$$\begin{aligned} D(x^*) &\leq d! h^d \left(\frac{1}{\prod_{j=0}^d \left| \frac{1}{2n} - \frac{j}{n} \right|} + \frac{1}{\prod_{j=1}^{d+1} \left| \frac{1}{2n} - \frac{j}{n} \right|} \right) \\ &= n 2^{d+1} d! \left(\frac{2d+1}{\prod_{j=1}^{d+1} (2j-1)} + \frac{1}{\prod_{j=1}^{d+1} (2j-1)} \right) = n \frac{2^{2d+2}}{\binom{2d+1}{d}}. \end{aligned}$$

□

Proposition 1. *If $d = 1$, then the Lebesgue constant associated to rational interpolation with the basis functions $b_j(x)$ in (1) at equidistant nodes satisfies*

$$\Lambda_n \geq a_n \ln(n) + b_n,$$

where $\lim_{n \rightarrow \infty} a_n = 2/\pi$ and $\lim_{n \rightarrow \infty} b_n = 0$.

Proof. If $d = 1$, then the weights β_j in (2) simplify to

$$\beta_j = \begin{cases} 1, & \text{if } j = 0, \\ 2, & \text{if } 1 \leq j \leq n-1, \\ 1, & \text{if } j = n. \end{cases} \quad (8)$$

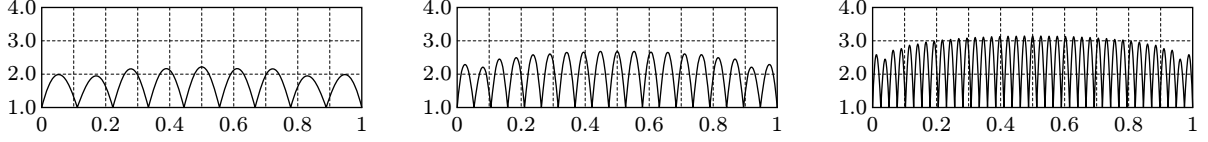


Figure 2: Lebesgue function of the Floater–Hormann interpolants for $d = 1$ at $n + 1$ equidistant nodes for $n = 9, 19, 39$.

Assume first that $n = 2k + 1$ is odd. The proof is very similar to the one in Theorem 2, except that we use $x^* = 1/2$. According to our numerical experiments, this is where the maximum of the Lebesgue function appears to occur (see Figure 2).

We first derive a lower bound for the denominator $D(x^*)$. Due to the symmetry of $D(x)$ with respect to x^* , the first and the last $k + 1$ terms in the sum are equal and so

$$D(x^*) = \left| \sum_{j=0}^n \frac{(-1)^j \beta_j}{x^* - x_j} \right| = 2n \left| \sum_{j=0}^n \frac{(-1)^j \beta_j}{n - 2j} \right| = 4n \left| \sum_{j=0}^k \frac{(-1)^j \beta_j}{2k + 1 - 2j} \right| = 4n \left| \sum_{j=0}^k \frac{(-1)^j \beta_{k-j}}{2j + 1} \right|.$$

Now using the triangle inequality, Equation (8), and the fact that the Leibniz series converges to $\pi/4$ with

$$\left| \sum_{j=0}^{k-1} \frac{(-1)^j}{2j + 1} - \frac{\pi}{4} \right| \leq \frac{1}{2k + 1},$$

we have

$$\begin{aligned} D(x^*) &\leq 4n \left(\left| \sum_{j=0}^{k-1} \frac{(-1)^j \beta_{k-j}}{2j + 1} \right| + \left| \frac{(-1)^k \beta_0}{2k + 1} \right| \right) = 8n \left| \sum_{j=0}^{k-1} \frac{1}{2j + 1} \right| + \frac{4n}{2k + 1} \\ &\leq 8n \left(\frac{\pi}{4} + \frac{1}{2k + 1} \right) + 4 = 2n\pi + 12. \end{aligned} \quad (9)$$

It remains to find a lower bound for the numerator $N(x^*)$. With the same arguments as above, it follows that

$$N(x^*) = 2n \sum_{j=0}^n \frac{\beta_j}{|n - 2j|} = 4n \sum_{j=0}^k \frac{\beta_j}{2k + 1 - 2j} = 8n \sum_{j=0}^{k-1} \frac{1}{2j + 1} + \frac{4n}{2k + 1} \geq 4n \ln(n) + 4,$$

and together with (9) we obtain

$$\Lambda_n \geq \frac{2}{\pi + 6/n} \ln(n) + \frac{2}{n\pi + 6}.$$

Finally, if $n = 2k$ is even, then the point $x = 1/2$ is a node and the Lebesgue function equals one there. Referring to Figure 1, we consider $x^* = 1/2 + 1/(2n)$ instead in this case. Applying the same reasoning as for odd n leads to

$$D(x^*) = 2n \left| 4 \sum_{j=0}^{k-1} \frac{(-1)^{k+j}}{2j + 1} + \frac{1}{2k - 1} + \frac{1}{2k + 1} \right| \leq 2n\pi + \frac{2n}{n - 1} + \frac{10n}{n + 1}$$

and

$$N(x^*) = 2n \left(4 \sum_{j=0}^{k-1} \frac{1}{2j + 1} - \frac{1}{2k - 1} + \frac{1}{2k + 1} \right) \geq 4n \ln(n + 1) - \frac{2n}{n - 1} + \frac{2n}{n + 1},$$

hence

$$\Lambda_n \geq \frac{2}{\pi + \frac{6n-4}{n^2-1}} \ln(n + 1) + \frac{2}{\pi(n^2 - 1) + 6n - 4},$$

which concludes the proof. \square

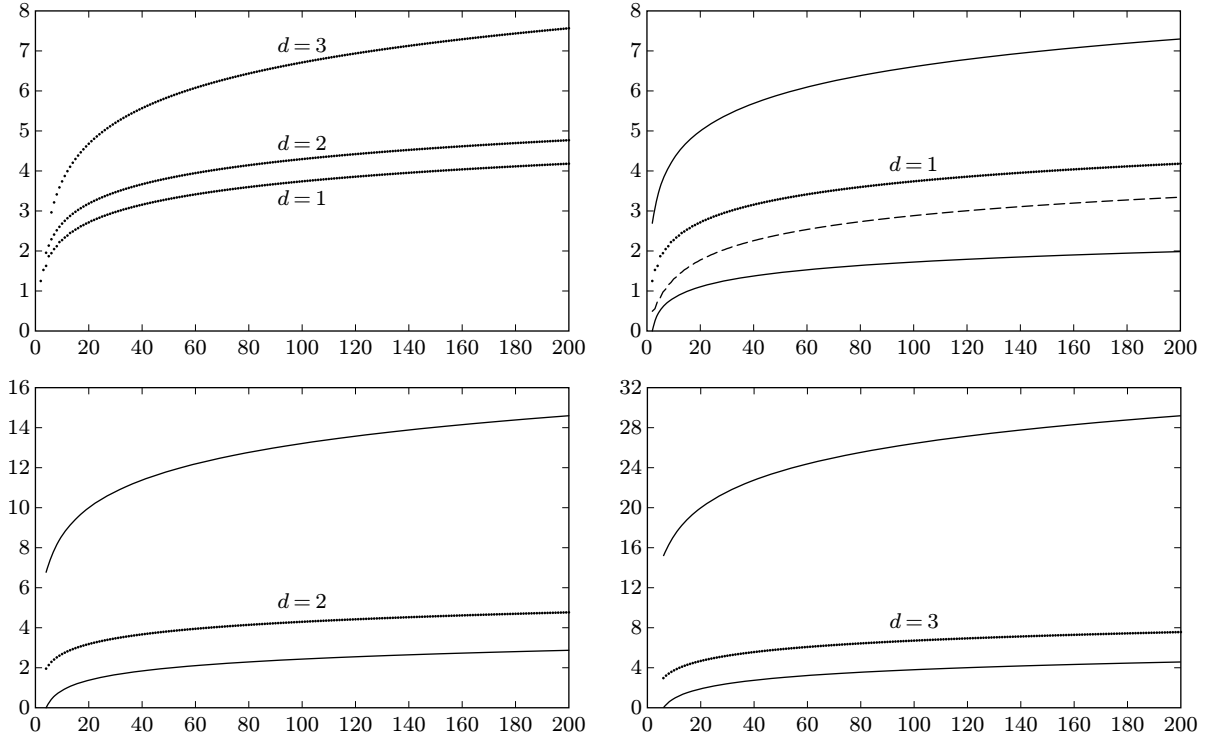


Figure 3: Comparison of the Lebesgue constants of the Floater–Hormann interpolants at $n + 1$ equidistant nodes for $2d \leq n \leq 200$ and $d = 1, 2, 3$ to each other (top left) and to the upper and lower bounds in Theorems 1 and 2. For $d = 1$, the improved lower bound in Proposition 1 is shown by the dashed curve (top right).

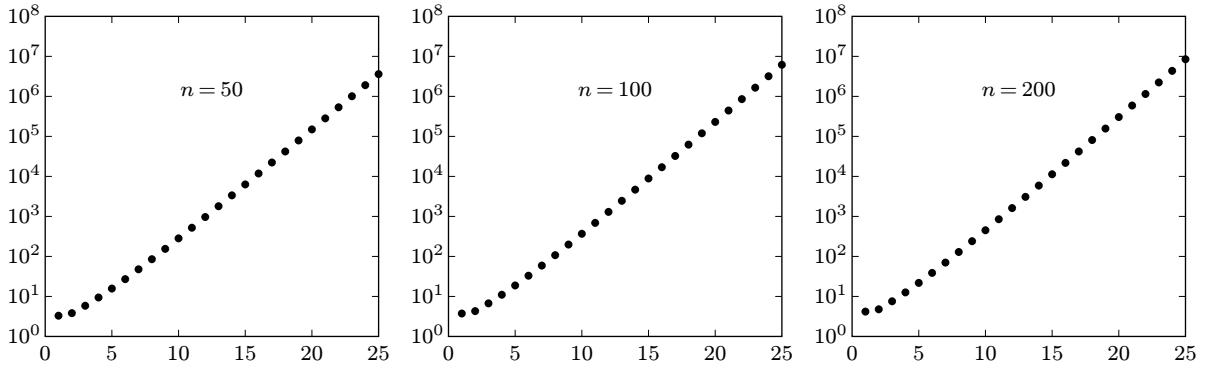


Figure 4: Lebesgue constants of the Floater–Hormann interpolants at a fixed number of $n + 1$ equidistant nodes for $1 \leq d \leq 25$.

4 Numerical experiments

We performed numerous experiments to verify numerically that the behaviour of the Lebesgue constant associated to the family of barycentric rational interpolants is as predicted by the theoretical results in the previous sections. Figure 3 confirms that the growth of Λ_n is logarithmic in the number of interpolation nodes. These results further suggest that for fixed d the coefficient $\binom{2d+1}{d}/2^{d+2}$ of the logarithmic term in our lower bound in Theorem 2 is a better estimate of the true value than the factor

2^{d-1} in our upper bound in Theorem 1. However, both factors indicate that for fixed n the growth of the Lebesgue constant with respect to d is exponential, which is confirmed by the examples in Figure 4. Finally, Figure 1 shows that this exponential growth seems to always happen near the boundary of the interpolation interval, whereas the behaviour of the Lebesgue function away from the boundary is almost independent of d . This suggests to consider distributions of nodes which are uniform in the centre and clustered towards the boundary, and we plan to study this setting in a forthcoming paper.

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