Hybridized Discontinuous Galerkin for Nonlinear Elasticity

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HDG formulations

- The HDG formulations are derived from minimization of the potential energy of the system among both the deformation and the numerical fluxes.
- Such derivation guarantees symmetric stiffness matrix through construction.
- The deformation and its gradient converge with order k+1 when polynomial of degree k are adopted for both the fields.

KEY IDEAS

Discrete variational principle

- Given conforming mesh $\mathcal{T}_h$ over the domain $\mathcal{B}_0$, Γ is the set of element boundaries.
- Solution renders stationary point of $I_h[\varphi^h, \hat{\varphi}^h]$

$$I_h[\varphi^h,\hat{\varphi}^h] = \sum_{K\in\mathcal{T}_h}\int_K W(\mathbf{D_{DG}}(\varphi^h,\hat{\varphi}^h)) - \mathbf{f}\cdot\varphi^h \; d\Omega - \int_{\partial K\in\Gamma^N} \mathbf{T}\cdot\hat{\varphi}^h \; dS + \frac{\tau}{2}\int_{\partial K\in\Gamma}(\varphi^h-\hat{\varphi}^h)\cdot(\varphi^h-\hat{\varphi}^h)\; dS$$

Derivatives with HDG formulations

- Lifting Operator:

$$(\mathbf{R}(v),\mathbf{w})_K = (v, \mathbf{w}\cdot\mathbf{N}^K)_{\partial K}$$

Here N^K is the unit outward normal to the element boundary.

- DG Derivative:

$$\mathbf{D_{DG}}(\varphi^h,\hat{\varphi}^h) = \nabla\varphi^h + \mathbf{R}(\hat{\varphi}^h - \varphi^h)$$

Euler Lagrange equations

$$(\mathbf{P}^\pi,\delta\varphi^\mathbf{h})_K - (\hat{\mathbf{T}}^\pi,\delta\varphi^\mathbf{h})_{\partial K} = (\mathbf{f},\delta\varphi^\mathbf{h})_K$$

$$\sum_K(\hat{\mathbf{T}}^\pi,\delta\hat{\varphi}^h)_{\partial K} = \sum_{\partial K\in\Gamma^N}(\mathbf{T},\delta\hat{\varphi}^h)_{\partial K}$$

Where, $\mathbf{P} = \dfrac{\partial W}{\partial \mathbf{F}}$

$$(\mathbf{P},\mathbf{w})_K = (\mathbf{P}^\pi,\mathbf{w})_K$$

$$\hat{\mathbf{T}}^\pi = \mathbf{P}^\pi\cdot\mathbf{N}^K + \tau(\varphi^h - \hat{\varphi}^h)$$

Convergence Study

Model Problem

On unit square

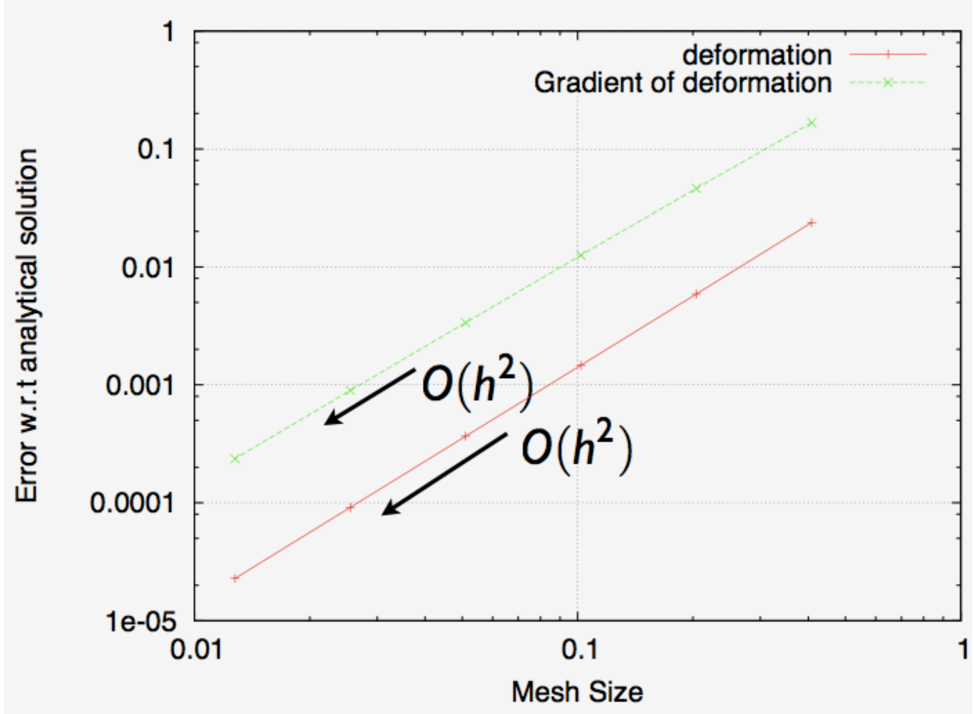
Affine approximations for both the fields

Neo-Hookean material :

$$W(\mathbf{F}) = \frac{\lambda}{2}(\log(\det\mathbf{F}))^2 - \mu\log(\det\mathbf{F}) + \frac{\mu}{2}\mathbf{F}:\mathbf{F}.$$

Exact Solution:

$$\varphi(X,Y) = X^2\,\hat{\mathbf{e}}_X + Y^2\,\hat{\mathbf{e}}_Y$$



Incompressible Limit

Growing Annulus

Neo-Hookean material with parameters: $E=3$ and $\nu = 0.499999$

Affine approximations are used for both the fields

Increase inner radius from R_0 to r_0

outer boundary, R_1, is kept traction free

Analytical Solution:

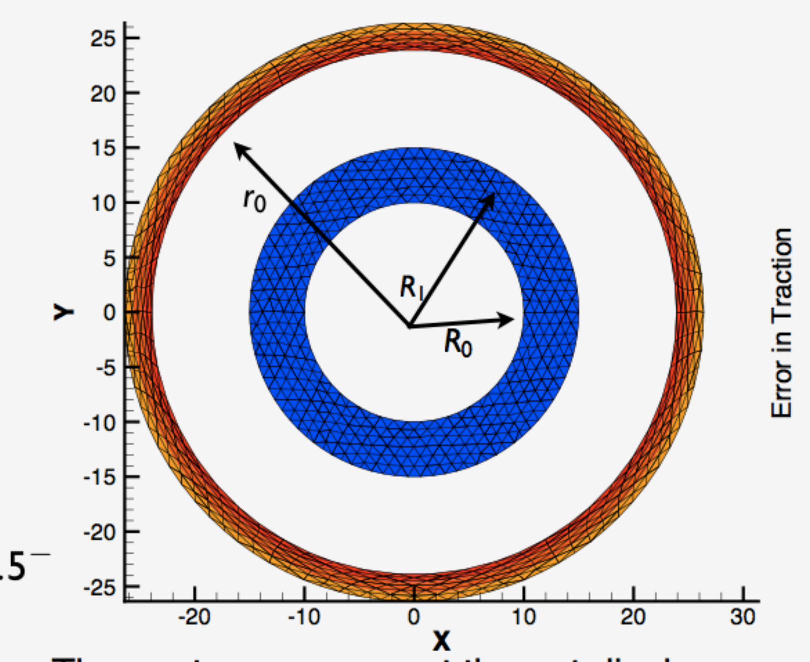
$$r(R) = \sqrt{r_0^2 + R^2 - R_0^2}$$

Traction in radial direction,

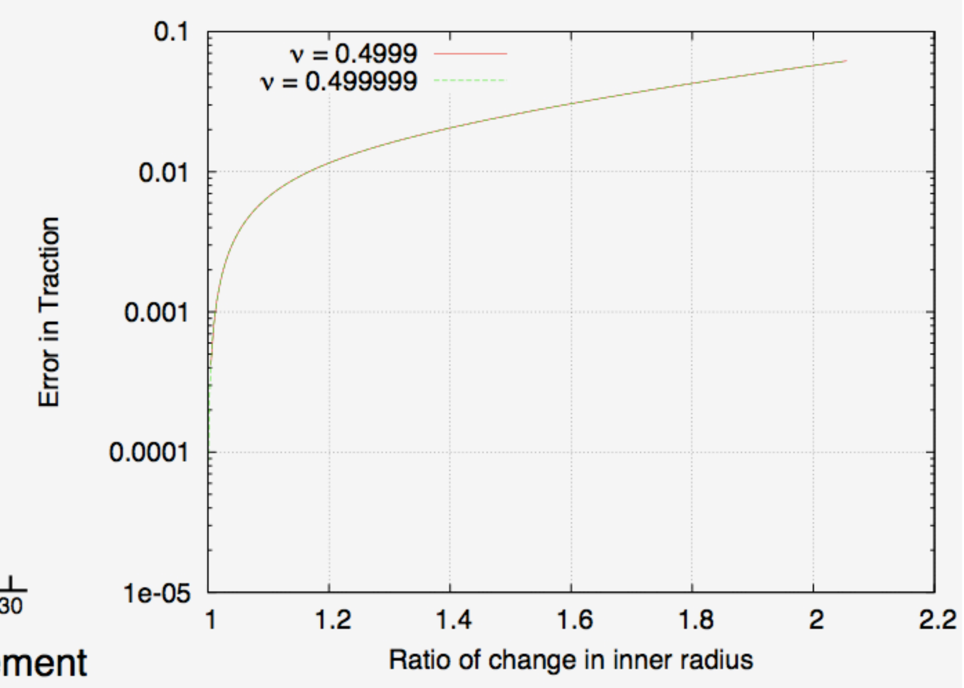
$$T_R(R) = \mathbf{e}_\mathbf{R}^\mathbf{T}\cdot\mathbf{P}\cdot\mathbf{e}_\mathbf{R} = p\frac{r}{R} + \left(\frac{R}{r} - \frac{r}{R}\right)$$

Here, p is partial pressure.

- No significant change in the traction as $\nu \to 0.5^-$
- Hence, no **'mesh locking'** observed for affine approximations.



The contours represent the net displacement

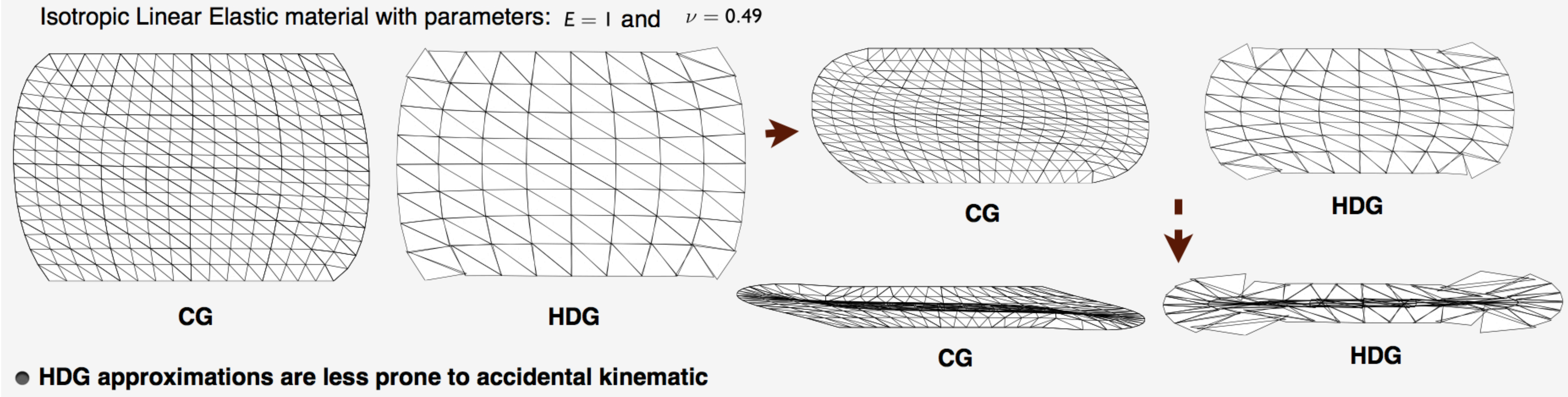


Mesh Based Kinematic Constraints

Square block under compression

Compare approximations for the deformation from CG and HDG formulations.

Isotropic Linear Elastic material with parameters: $E=1$ and $\nu = 0.49$



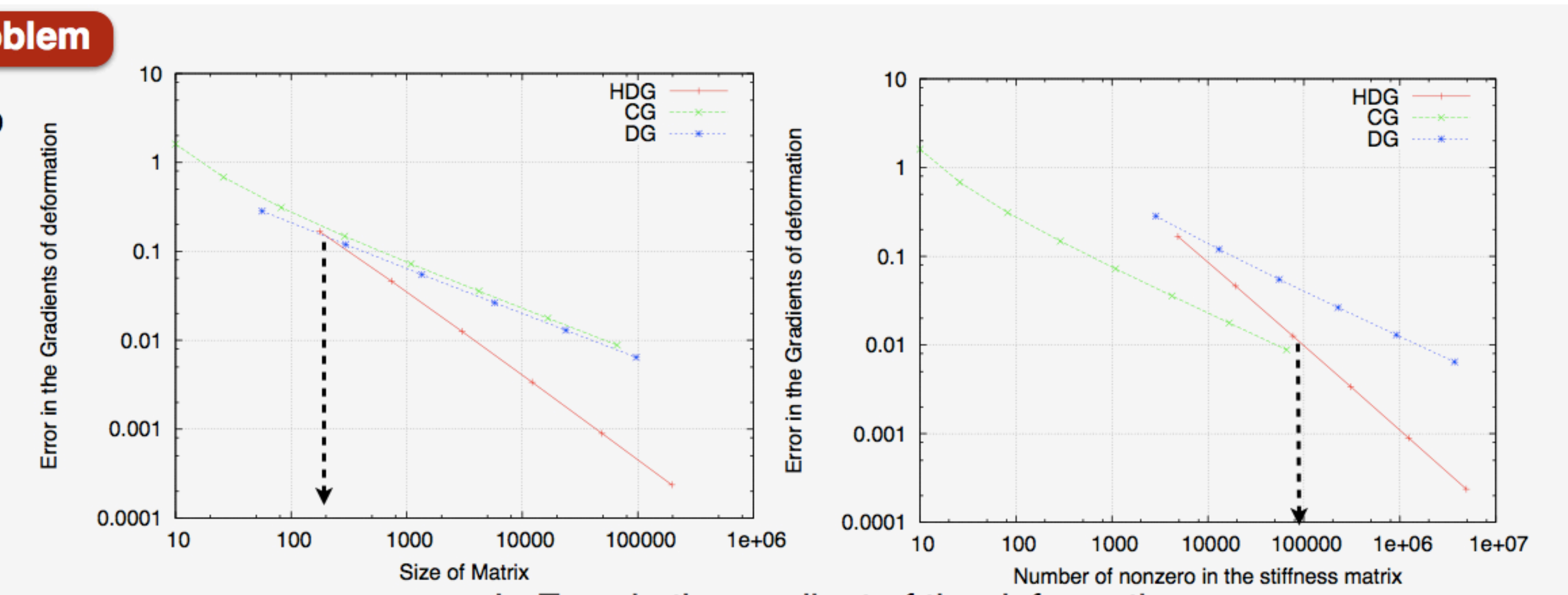
- **HDG approximations are less prone to accidental kinematic constraints imposed by the mesh.**

Efficiency

Compare CG, DG and HDG methods for Model Problem

Computational efficiency is taken to be proportional to size of the stiffness matrix or the number of nonzero entries in the stiffness matrix.

- **The HDG methods show better efficiency for the accuracy in the gradient of the deformation compare to the CG and the DG methods as the number of DOF increase.**



L_2 Error in the gradient of the deformation

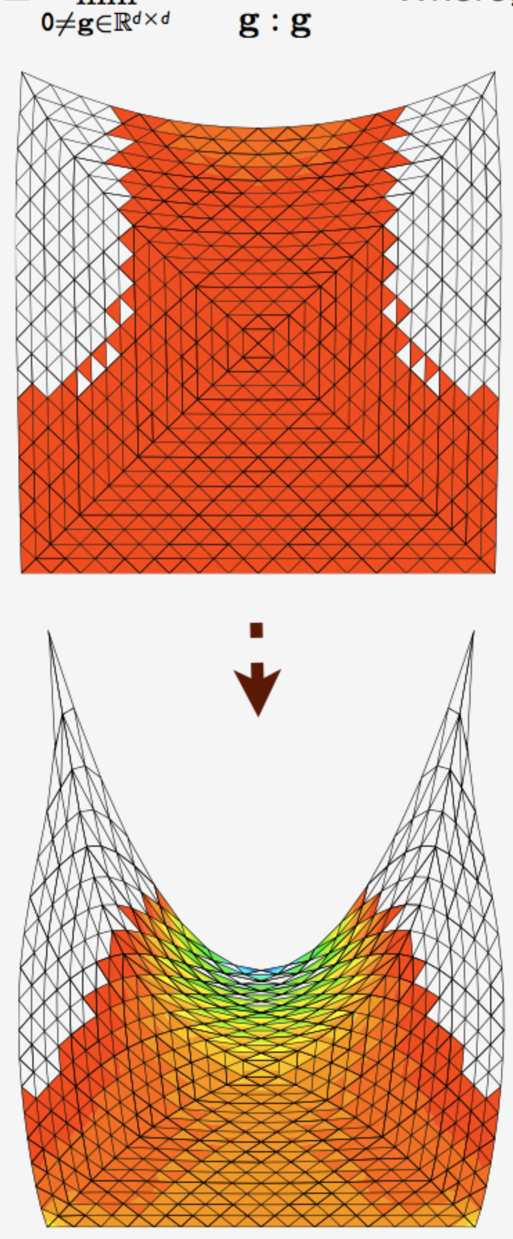
Instabilities

Indentation of square block

Neo-Hookean material

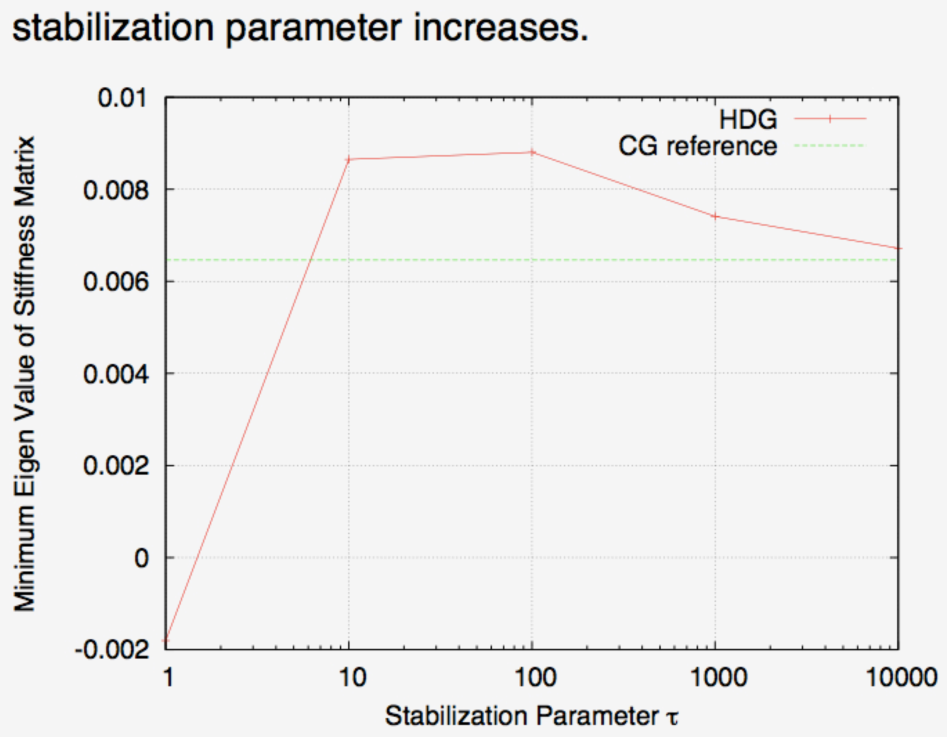
Lower surface held fixed, upper is mapped to a parabola

$$\lambda_{min}(\mathbf{X}) = \min_{0\neq\mathbf{g}\in\mathbb{R}^{d\times d}}\frac{\mathbf{g}:\mathbb{A}(\mathbf{F}):\mathbf{g}}{\mathbf{g}:\mathbf{g}} \quad \text{Where, } \mathbb{A} = \frac{\partial^2 W}{\partial\mathbf{F}^2}$$



Regions with -ve minimum eigen values are colored.

Comparing minimum eigen value of the global stiffness matrix for the CG and the HDG methods as stabilization parameter increases.

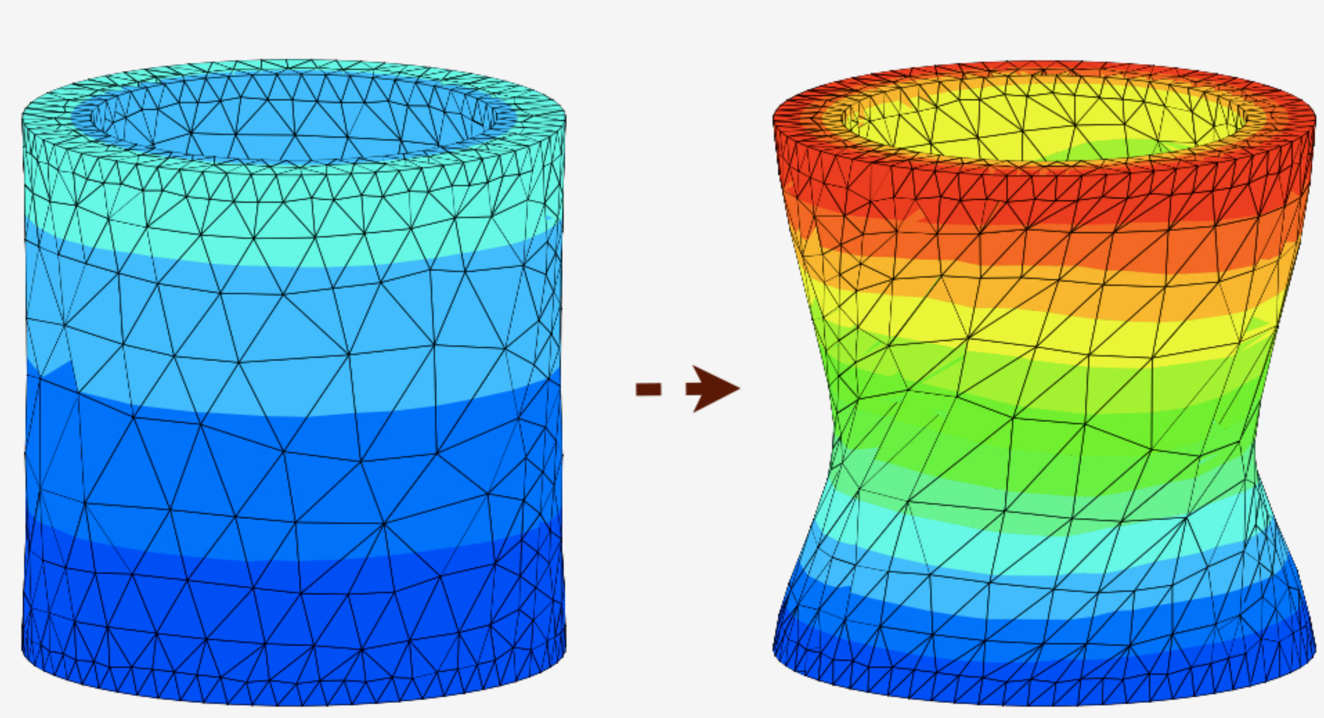


Large Deformations

Torsional load on a hollow cylinder

Neo-Hookean material

Upper surface rotated by 90° while lower surface held fixed.



The contours represent the net displacement field.

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