Barycentric Coordinates on Surfaces

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Barycentric coordinates good for:

- interpolation
- shading
- deformation
- Bezier surfaces
- parameterization
- interior distance
- image cloning
- shape retrieval
- finite elements
Goal

Define barycentric coordinates on surfaces:
- generalize planar
- intrinsic
- fast to compute
Definition on the plane

\[ p = \text{affine comb of vertices} \]

\[ p = \sum b_i v_i, \quad \text{where } \sum b_i = 1 \]
Challenges on surfaces

\[ p = \sum b_i v_i, \quad \text{where} \quad \sum b_i = 1 \]

\( p \) belongs to convex hull of vertices \( \times \)
Challenges on surfaces

\[ p = \sum b_i v_i, \quad \text{where} \quad \sum b_i = 1 \]

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Type of Triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mobius, Alfeld et al, Cabral et al</td>
<td>Spherical triangles</td>
</tr>
<tr>
<td>Ju et al</td>
<td>Spherical convex</td>
</tr>
<tr>
<td>Langer et al</td>
<td>Spherical all</td>
</tr>
</tbody>
</table>
Challenges on surfaces

\[ p = \sum b_i v_i, \quad \text{where} \quad \sum b_i = 1 \]

involves Cartesian coords

not intrinsic
Our approach

\[ p = \sum b_i v_i, \quad \text{where} \quad \sum b_i = 1 \]

\[ p = \text{c.mass} \{ v_i \} \text{ with masses } \{ b_i \} \]
Our approach

\[ p = \sum b_i v_i, \quad \text{where} \quad \sum b_i = 1 \]

\[ p = \text{c.mass \{v}_i\} \quad \text{with masses \{b}_i\} \]
Outline

- Introduction
- Riemannian Center of Mass
- Construction
- Computation
- Properties
- Results
- Applications
Usual center of mass

\[ p = \text{c.mass } \{v_i\} \text{ with masses } \{b_i\} \]

\[ p = \arg \min_x \sum b_x d^2(x, v_i) \]

Euclidean distance
Riemannian center of mass

\[ p = \text{c.mass } \{v_i\} \text{ with masses } \{b_i\} \]

\[ p = \arg \min_x \sum b_i d^2(x, v_i) \]

Geodesic distance
Karcher’s theorem

If the points are not “too far” from each other, the Riemannian center of mass is unique

\[ U(x) = \sum b_i d^2(x, v_i) \]

- \( U(x) \) has a unique minimum,
- which is the unique zero of \( \nabla U(x) \)
Construction

\( p \) has barycentric coordinates \( \{ b_i \} \)

\( p = \text{c.mass} \ \{ v_i \} \) with masses \( \{ b_i \} \)

\( \nabla U(x) = \vec{0} \) must hold at \( x = p \)
Construction

\[ \vec{0} = \nabla U(x) \bigg|_{x=p} \]

\[ = \nabla \left( \sum b_i d^2 (x, v_i) \right) \bigg|_{x=p} \]

\[ = \sum b_i \nabla d^2 (x, v_i) \bigg|_{x=p} \]

\[ = \sum b_i \vec{g}_i \]
Construction

\[ \sum b_i \vec{g}_i = \vec{0} \approx \vec{p} \]

\[ \sum b_i = 1 \]
Construction

\[ \sum b_i \vec{g}_i = \vec{0} \simeq p \]

\[ \sum b_i = 1 \]

\[ \{b_i\} = \text{planar baryc coords of } p \text{ wrt dashed polygon} \]
Computation Summary

- Pick a kind of planar baryc coords
- E.g. Mean Value Coords

- Compute the gradient polygon at each point
- Explicit formula exists

- Surface baryc coords = planar baryc coords wrt gradient polygon
Computation of gradients

inverse exponential map

Schmidt et al. [2006]
Computation of gradients

$$\vec{g}_i = \nabla d^2(x, v_i)|_{x=p} \simeq \exp_p^{-1}(v_i)$$

Schmidt et al. [2006]
Intuitiveness

I point towards v1. My length is equal to geod. dist. from p to v1
Intuitiveness

I am at the “correct” distance and direction wrt p

I point towards v1. My length is equal to geod. dist. from p to v1
Discrete setting
Discrete setting
Discrete setting
Discrete setting

Pentagon
Need five instances of “single source, all destinations”
Properties

- Defining properties
  - Lagrangian
  - Partition of unity
  - Riemannian center of mass

- Unique reconstruction from coordinates
  - Due to Karcher’s theorem

- Planar reproduction
  - If surface is plane, get planar coordinates back

- Similarity invariance

- Smoothness
Properties

- Edge linearity
Properties

- Isometry invariance

- Isometry invariance + unique reconstruction = isometry map can be reconstructed
Results

- Darkest red point – vertex wrt which coords are computed
- Dark blue – small value
- Dark red – large value
- Equally spaced isolines

- Planar baryc coords =
  =Mean Value Coordinates
Effect of surface shape
Variety
Variety
Effect of planar coordinate

Mean Value Coordinates

Maximum Entropy Coordinates
## Timing (seconds)

| $|V|$ | $|F|$ | Preprocess | Per-point |
|----|----|------------|------------|
| 2K | 4K | 0.81       | 0.0018     |
| 4K | 7K | 1.25       | 0.0018     |
| 8K | 16K| 5.46       | 0.0019     |
| 17K| 35K| 7.12       | 0.0021     |
| 27K| 56K| 12.71      | 0.0024     |
| 53K| 105K| 28.04     | 0.0027     |
Application: Interpolation

values \( \{f_i\} \) on vertices \( \{v_i\} \), define \( f(p) = \sum f_i b_i(p) \)
Application: Decal mapping

Local parameterizations – used for texturing
We use the same idea as in image warping

\[
q = \begin{bmatrix} U \\ V \end{bmatrix} = b_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
Application: Decal mapping
Based on isometry reconstruction property
Correspondence is exact, if true isometry
App: Correspondence Refinement

\[ \begin{align*} \{b_i\} \text{ wrt } \{v_i\} & \quad \Rightarrow \quad p' = \arg \min_x \sum b_i d^2(x, u_i) \end{align*} \]
App: Correspondence Refinement
Summary

Definition and construction of barycentric coordinates on surfaces:

- properly generalize existing planar coordinates
- insensitive to isometric deformations
- easy to implement, and fast to compute
Future work

- Better Karcher’s theorem
  - Other distances instead of geodesic in $U(x) = \sum m_i d^2(x, v_i)$
  - Uniqueness of c.m. for larger polygons?
- More empirical studies of various choices of
  - Distance
  - Planar barycentric coordinates
- Further applications of surface coordinates
Thank you

Software
- Szymon Rusinkiewicz for Trimesh2
- Danil Kirsanov for Exact Geodesic

3D models
- Daniela Giorgi
- AIM@SHAPE

Raison d'être
- Remy of Ratatouille whose posing for [Joshi et al 2007] got me interested in barycentric coordinates
Thank you

Thank You!