Recurrences and the Complexity of Divide and Conquer Algorithms

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Outline

- Analysis of recurrence expressions

Complexity Analysis

- **MergeSort**

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  - is this right?
  - can we generalize?

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- Our goal is to obain a closed-form formula for $T(n)$

Master Theorem

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with $a > 0$, $b > 1$, $d \geq 0$ the asymptotic complexity is

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Divide-and-Conquer Tree

$$T(n) = aT(n/b) + f(n)$$

size

n

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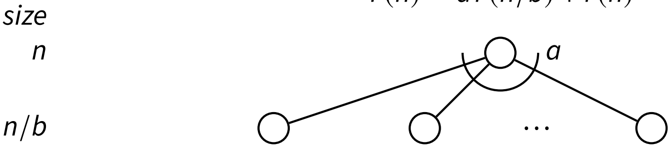

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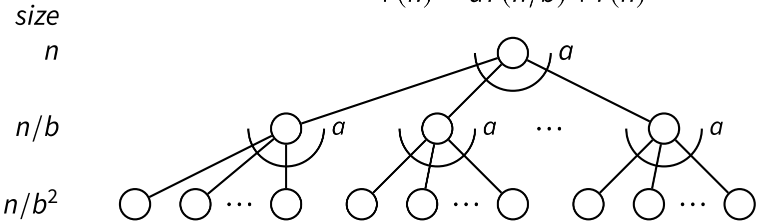

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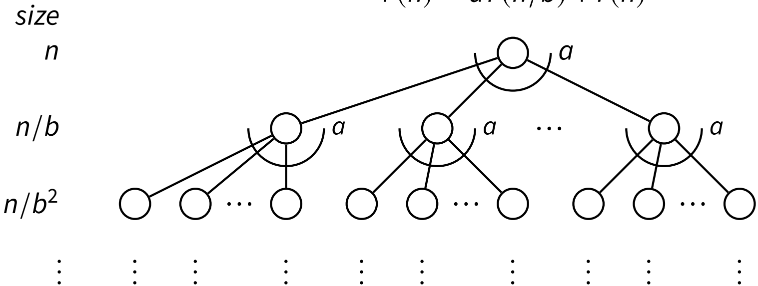

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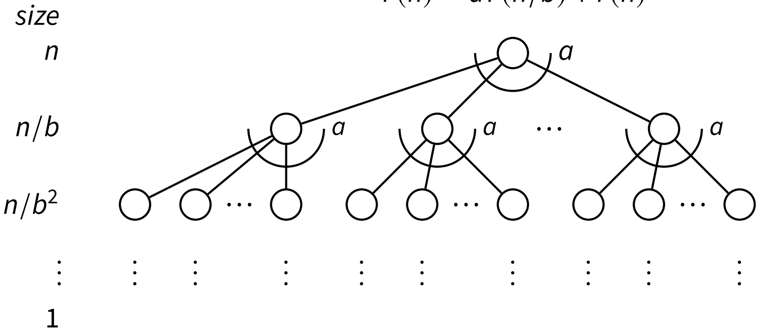

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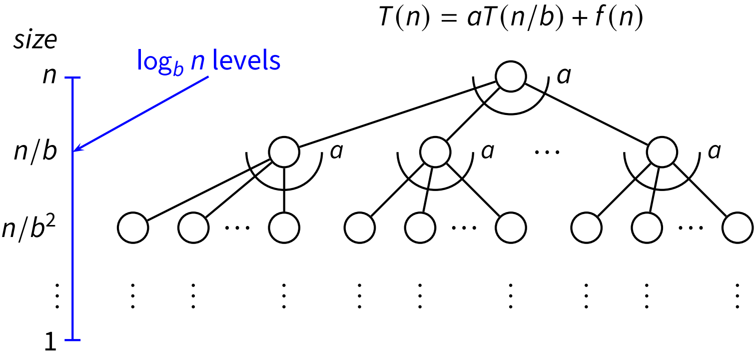

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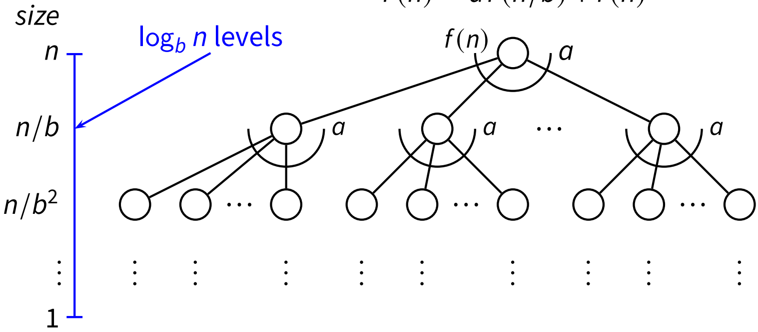

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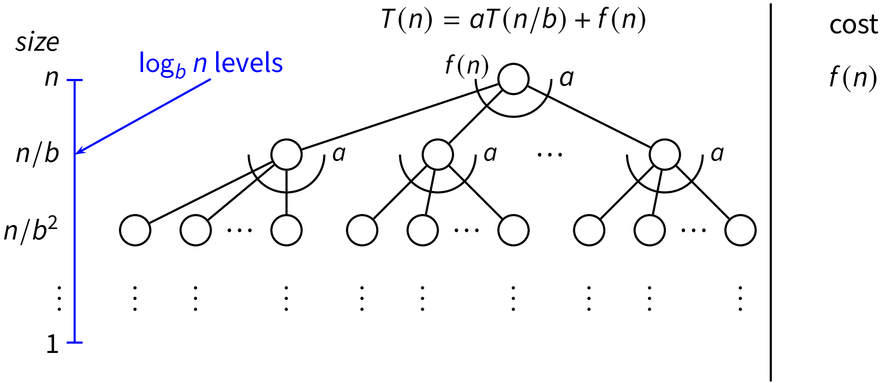

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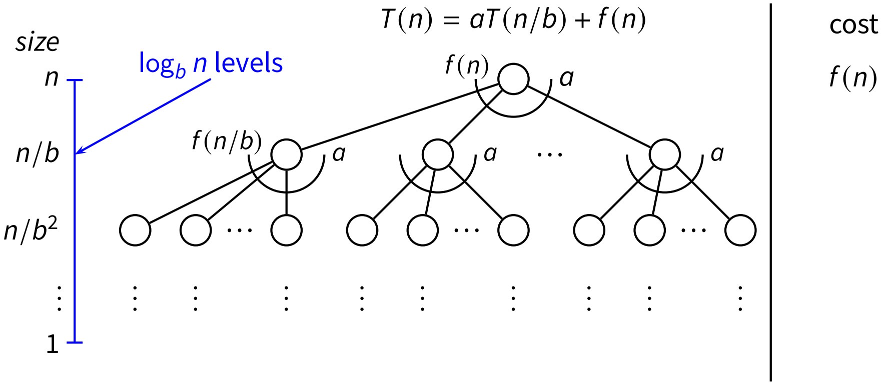

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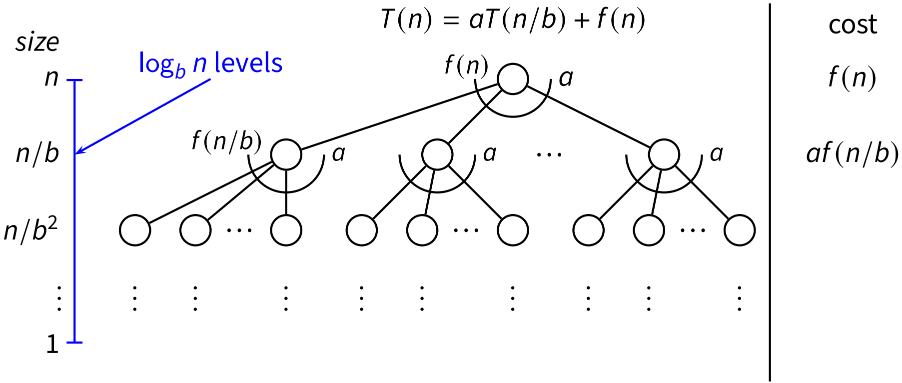

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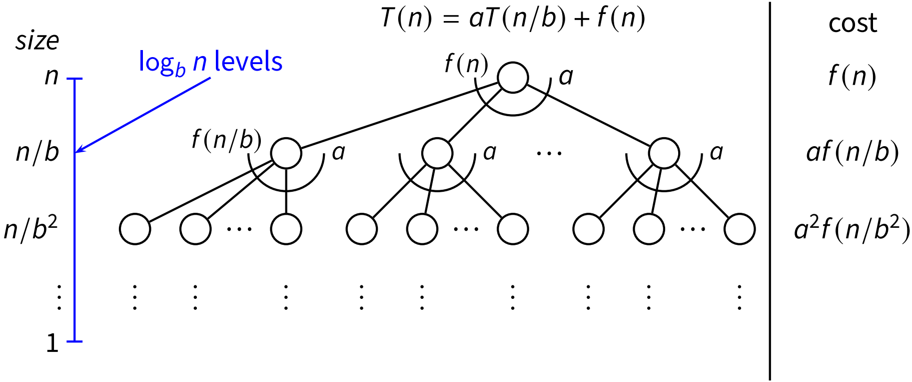

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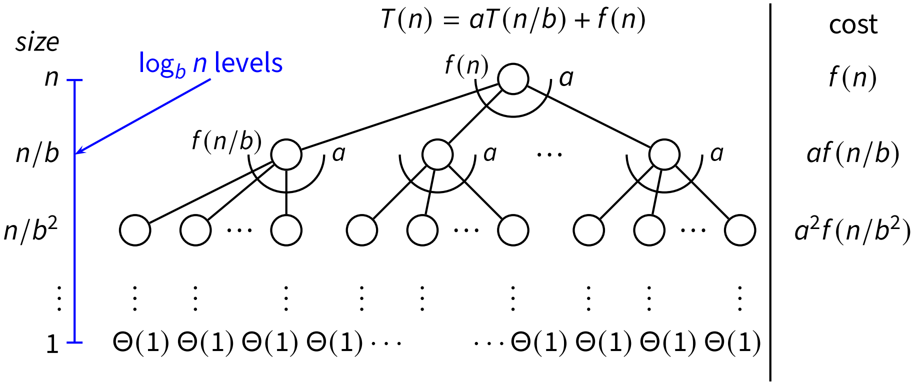

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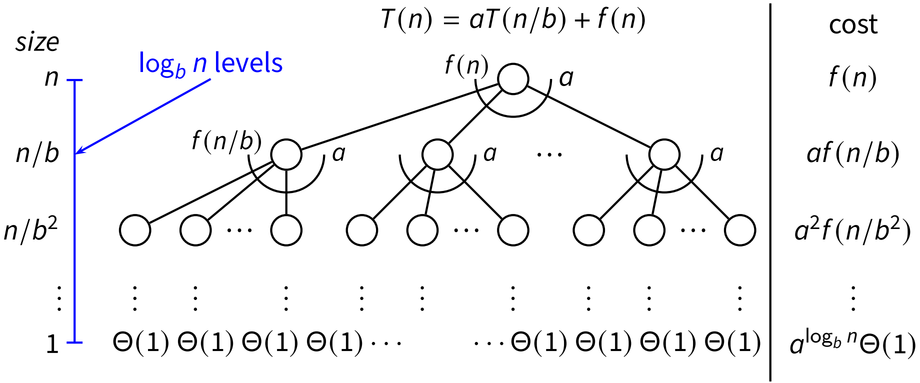

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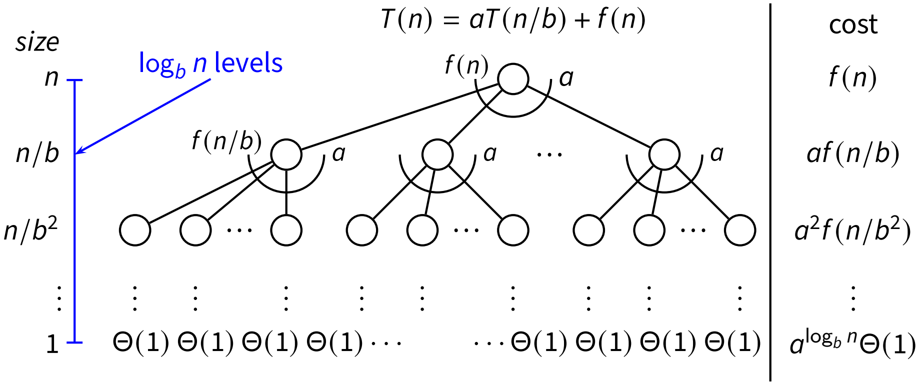

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- Let's look at the geometric-series component; we should figure out the asymptotic behavior of this term

$$O(n^d) \sum_{i=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

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$$g(\ell) = \begin{cases} \Theta(1) & \text{if } c < 1 \\ \Theta(\ell) & \text{if } c = 1 \\ \Theta(c^\ell) & \text{if } c > 1 \end{cases}$$

Divide-and-Conqer Complexity (4)

- Back to divide and conquer

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- In other words, we proved

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$