Randomized Algorithms

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Outline

- Examples
- Las Vegas and Monte Carlo algorithms
- More examples

Remember Quick-Sort?

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QUICKSORT(A, begin, end)
if begin < end
    q = PARTITION(A, begin, end)
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- Idea: *partition* the sequence $A[1 \dots n]$ in three parts
  - $A[1 \dots q-1]$ contain elements that are *less than or equal to* v
  - $A[q] = v$ is the “pivot” value ($v \in A$)
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$q = 6$

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$A[1 \dots q-1]$ $A[q+1 \dots n]$

A Classic Divide-and-Conquer

- ***Divide:*** partition A in $A[1 \ldots q-1]$ and $A[q+1 \ldots n]$ such that…

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- ***Combine:*** nothing to do here
  - it is all in the *partition* algorithm

```
PARTITION(A, begin, end)
q = begin
v = A[end]  // we deterministically choose v
for i = begin to end
    if A[i] ≤ v
        swap(A[i], A[q])
        q = q + 1
return q − 1
```

Complexity of QUICKSORT

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$$\boxed{T(n) = \Theta(n \log n)}$$

Complexity of QuickSort

QuickSort$(A, begin, end)$

if $begin < end$
  $q = $ **Partition**$(A, begin, end)$
  QuickSort$(A, begin, q-1)$
  QuickSort$(A, q+1, end)$

- *Worst case:* $q = 1$ or $q = n$ *not so improbable!*
  - the partition transforms P of size n in P of size $n-1$, so $T(n) = T(n-1) + \Theta(n)$

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A Randomized Solution

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- Simple

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RANDOMIZED-PARTITION(A, begin, end)
    i = RANDOM(begin, end)
    swap(A[i], A[end])
    return PARTITION(A, begin, end)
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Other Examples

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```
TREE-RANDOMIZED-INSERT(t, z)
if t = NIL
        return z
r = uniformly random value from {1, . . . , size(t) + 1}
if r = 1                              // Pr[r = 1] = 1/(size(t) + 1)
        size(z) = size(t) + 1
        return TREE-ROOT-INSERT(t, z)
if key(z) < key(t)
        left(t) = TREE-RANDOMIZED-INSERT(left(t), z)
else right(t) = TREE-RANDOMIZED-INSERT(right(t), z)
size(t) = size(t) + 1
return t
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- The idea is to make the algorithm run with their *good complexity* ***with high probability*** *on every input*
- The algorithm does not change that much, though

- However, we can do a lot more with randomized algorithms…

Simple Example

- Problem: *find a zero bit*
  - *Input:* an array A, of n bits, containing more or less the same number of 1s and 0s (*hamming weight* is roughly $n/2$)
  - *Output:* i such that $A[i] = 0$, or NIL if none exists

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- Obvious solution

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FIND-A-ZERO-BIT(A)
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        if A[i] == 0
                return i
return NIL
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- Problems?
  - what if A is sorted in reverse order? (all 1-bits before the 0-bits)
  - any *deterministic* search strategy is vulnerable

A Randomized Algorithm

A Randomized Algorithm

- Take one

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RANDOMIZED-FIND-A-ZERO-BIT1(A)
repeat
    i = RANDOM(1, |A|)
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- expected iterations: 2

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- Take two

```
RANDOMIZED-FIND-A-ZERO-BIT2(A)
for j = 1 to k
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- ▶ expected iterations: 2
- ▶ worst-case: unbounded!
- ▶ ***Las Vegas***

- Take two

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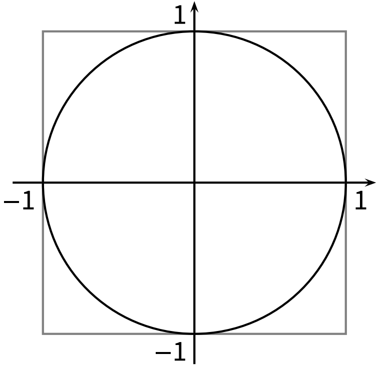
- ▶ ***Monte Carlo***

Simple Monte Carlo Algorithm

- Problem: *compute the surface of the unit disc*
  - you don't know the value of π—you may not even know that $S = \pi r^2$, but you know that a point

$$(x, y) \quad \text{is} \begin{cases} \textit{outside} \text{ the unit disc if} & x^2 + y^2 > 1 \\ \textit{inside} \text{ the unit disc if} & x^2 + y^2 \leq 1 \end{cases}$$



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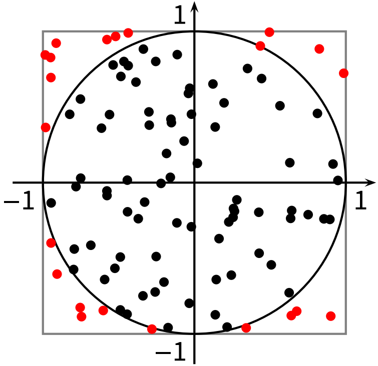

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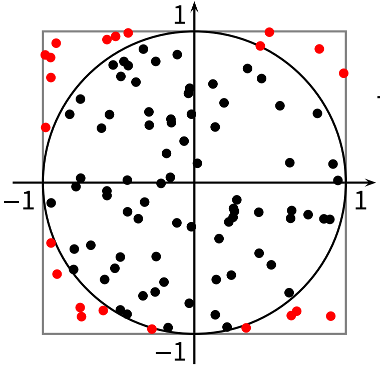

78 out of 100 points are *inside*

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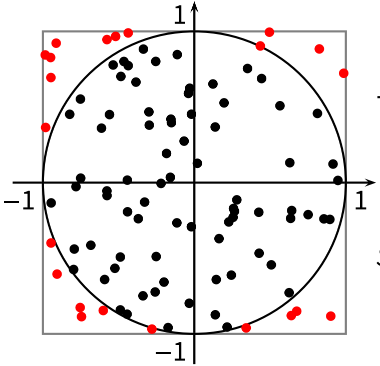

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$S = 4 \times 78/100 = 3.12$

Monte Carlo π

```
MONTE-CARLO-PI(n)
i = 0
for j = 1 to n
        x = RANDOM(−1, 1)
        y = RANDOM(−1, 1)
        if x^2 + y^2 ≤ 1
                i = i + 1
return 4i/n
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  - more specifically, the *expected* precision grows with n

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  - more specifically, the *expected* precision grows with *n*

- It is also easy to think about a better (adaptive) *stopping condition* other than going through *n* loops