Minimal Spanning Trees

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Outline

- MST problem
- Generic algorithm
- Prim and Kruskal

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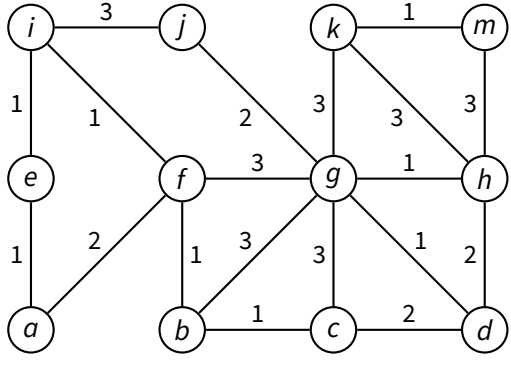
  - a ***spanning tree***
- T's total weight of the tree is minimal

$$w(T) = \sum_{(u,v) \in T} w(u, v)$$

  - a ***minimum-weight spanning tree***, or "minimum spanning tree"

Example



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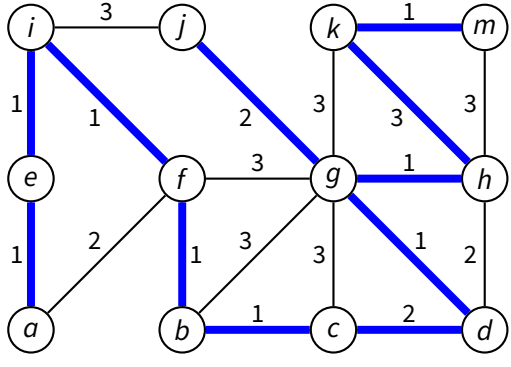

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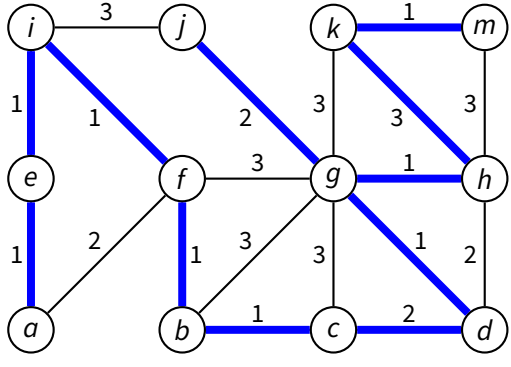

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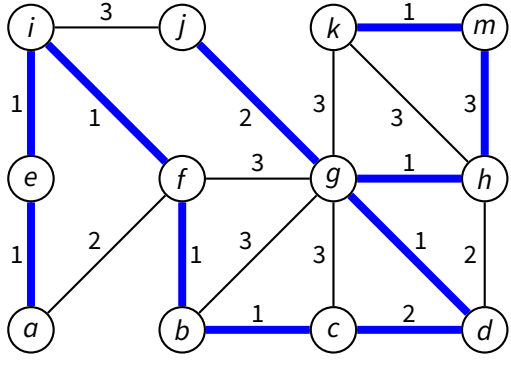

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- Does it work?

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  - more or less the *definition* of a greedy algorithm

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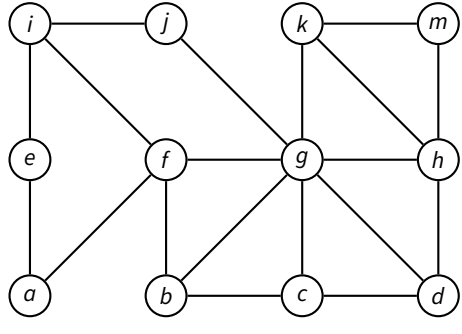

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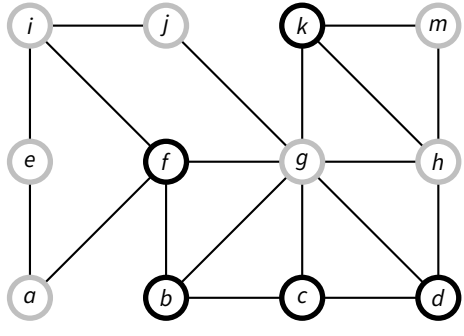

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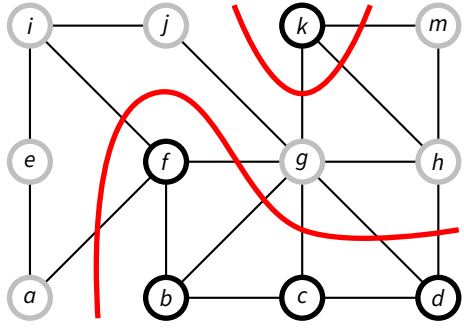

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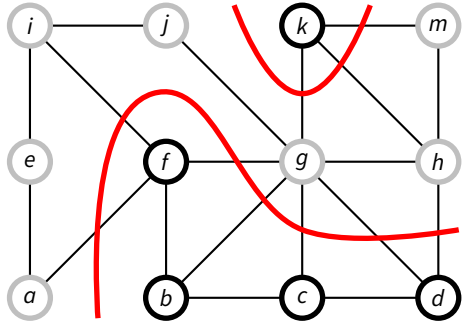

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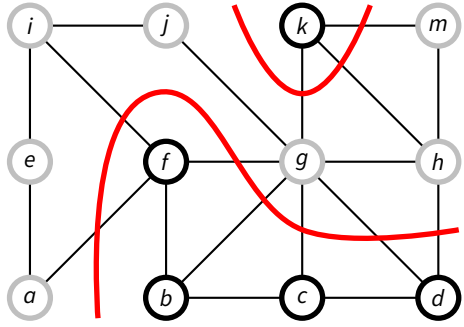

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- A cut $(S, V - S)$ *respects* a set of edges A if no edge in A crosses the cut

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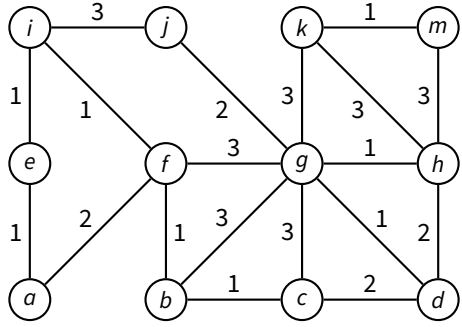

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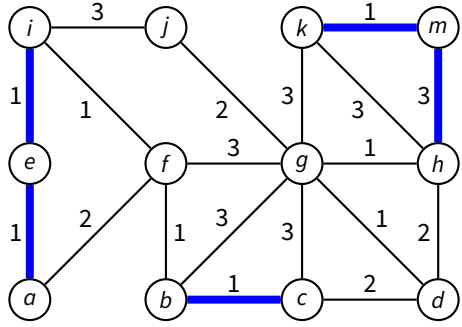

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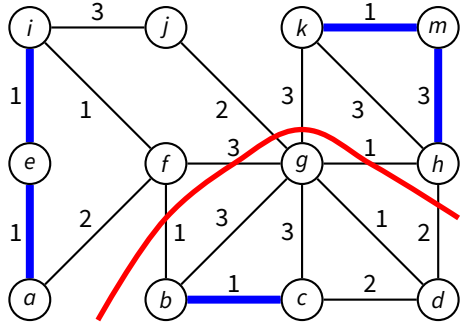

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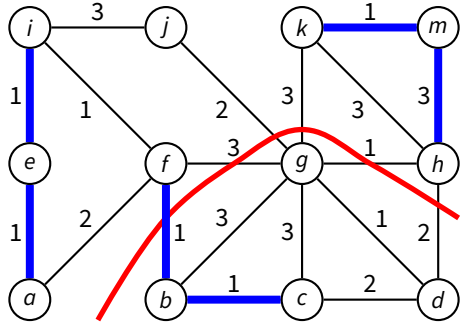

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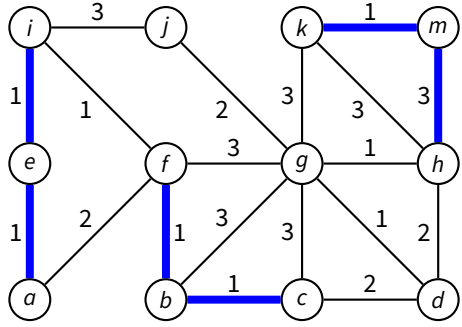

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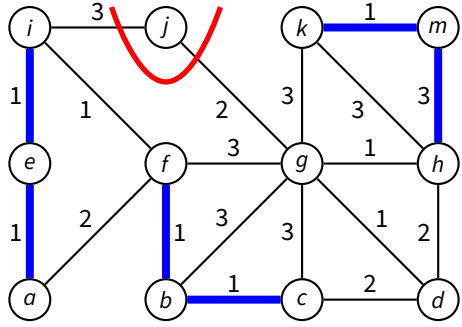

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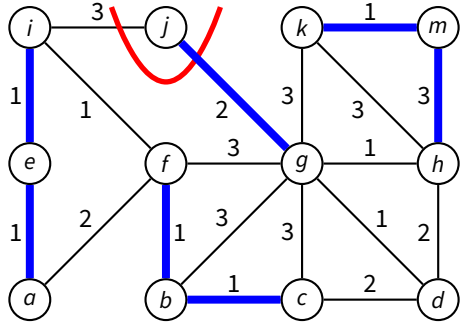

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  - incrementally builds a ***forest*** A

Concrete Algorithms

- Kruskal's algorithm (1956)
  - based on the generic minimum-spanning-tree algorithm
  - incrementally builds a ***forest*** A
- Prim's algorithm (1957)
  - based on the generic minimum-spanning-tree algorithm
  - incrementally builds a ***single tree*** A

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- *Union*(x, y) joins the sets containing x and y

Kruskal's Algorithm

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MST-KRUSKAL(G, w)
1  A = ∅
2  for each vertex v ∈ V(G)
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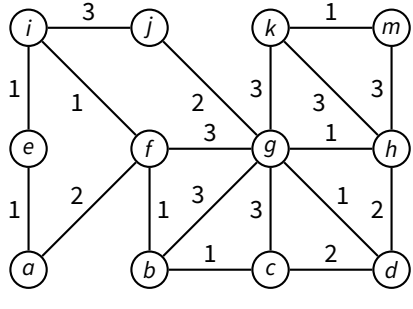

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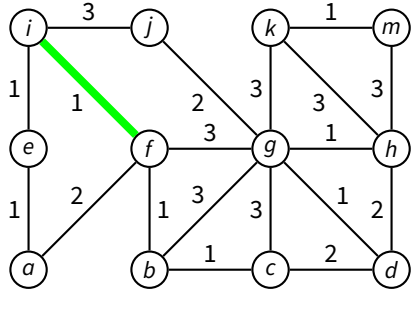

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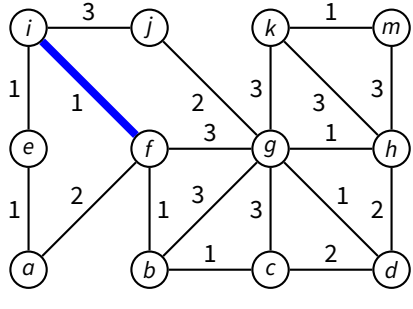

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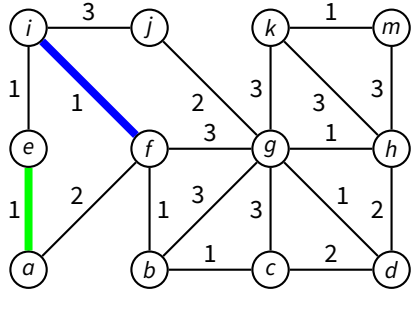

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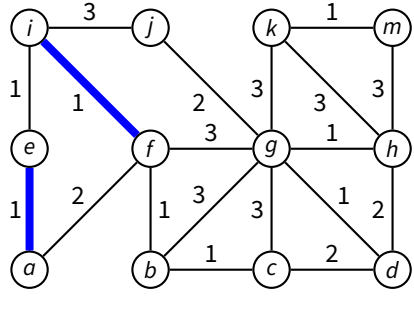

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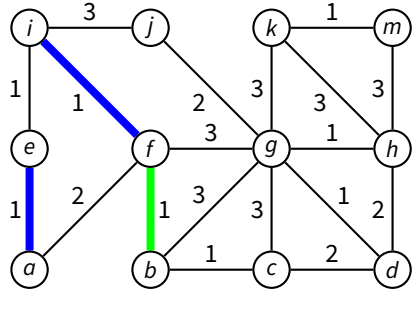

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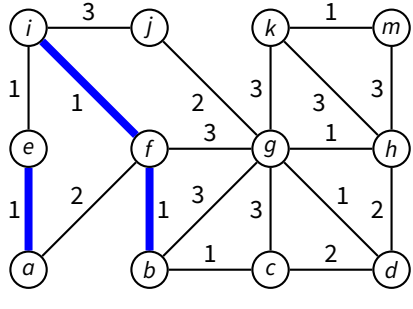

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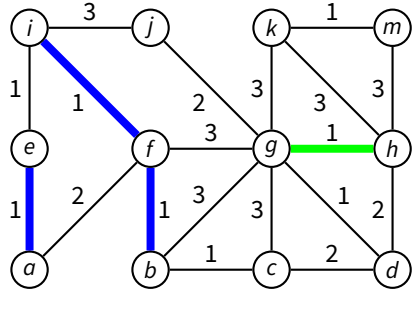

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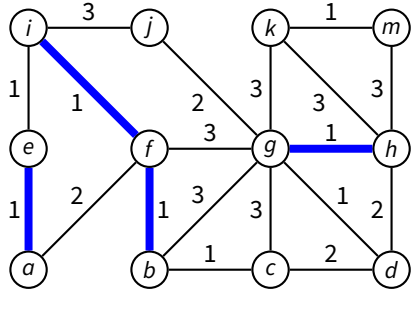

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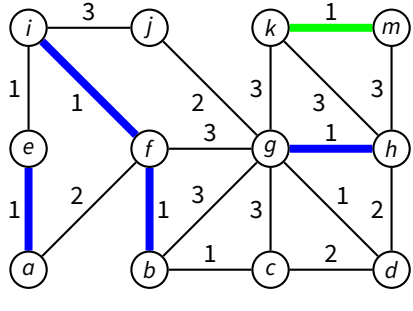

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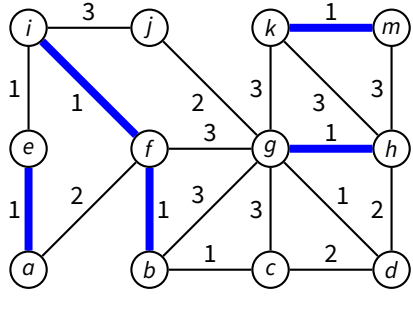

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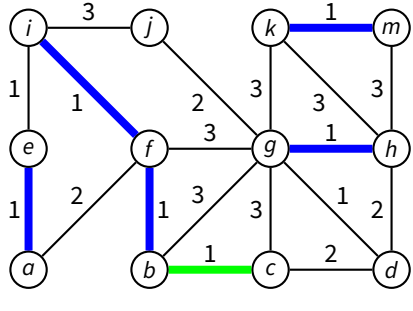

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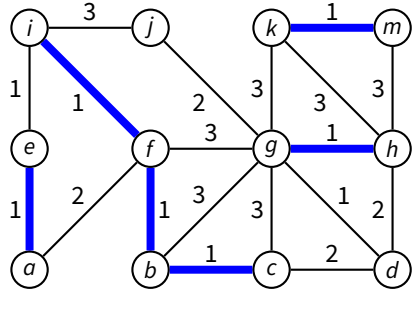

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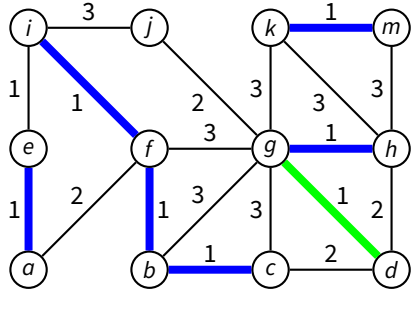

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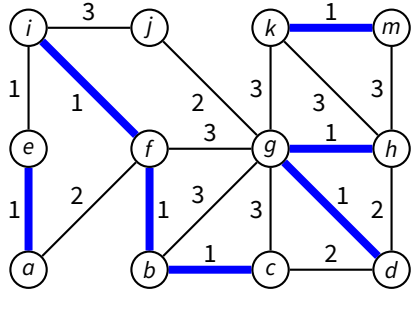

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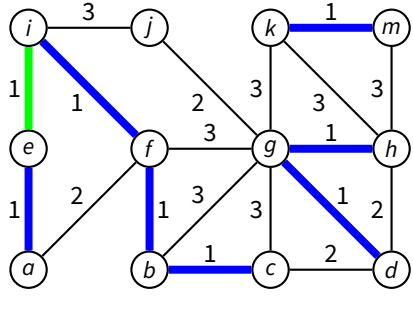

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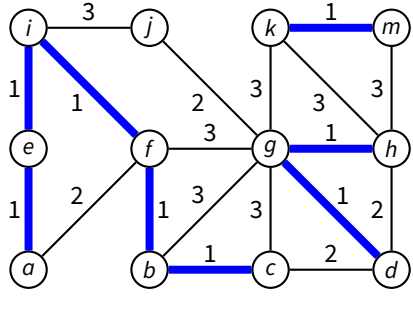

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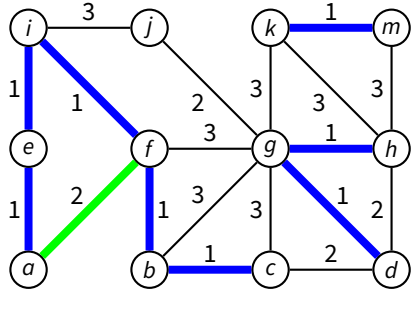

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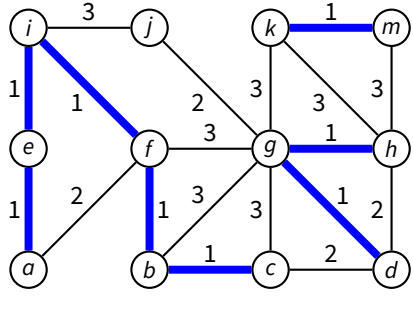

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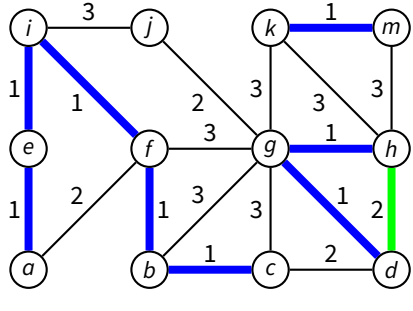

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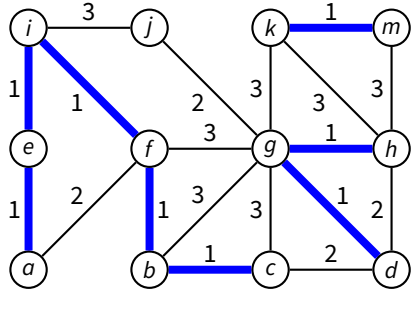

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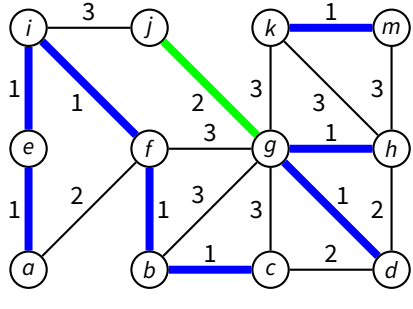

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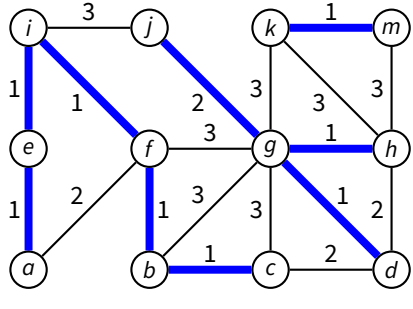

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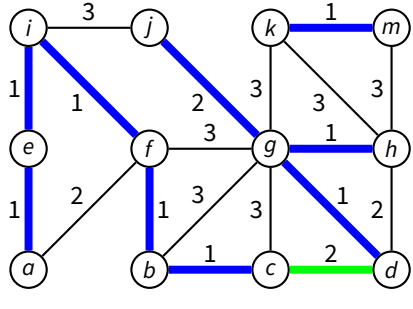

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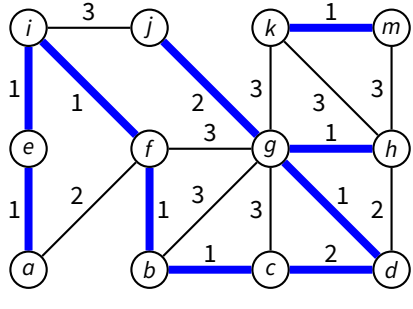

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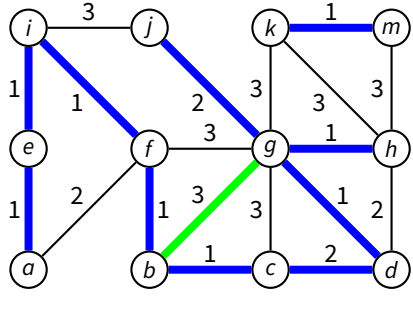

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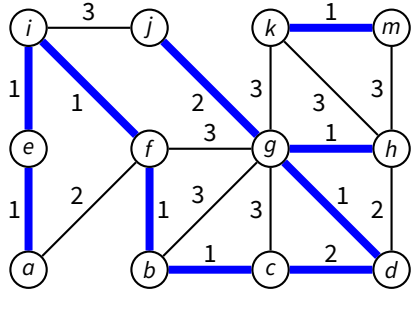

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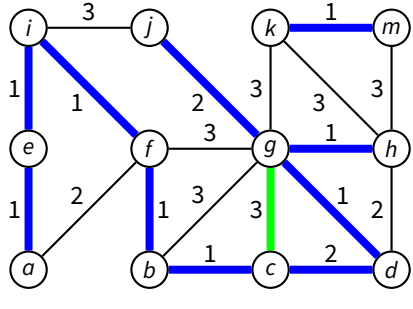

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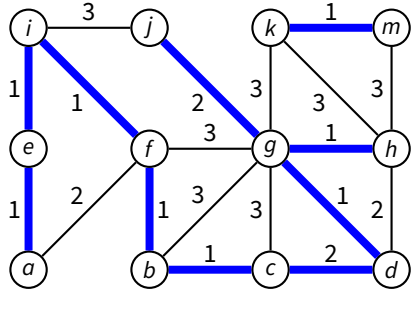

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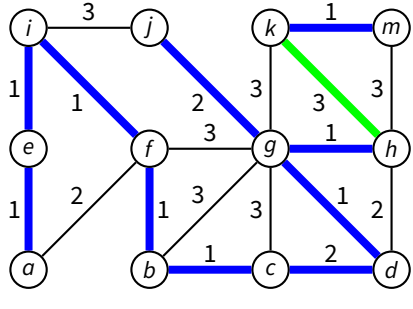

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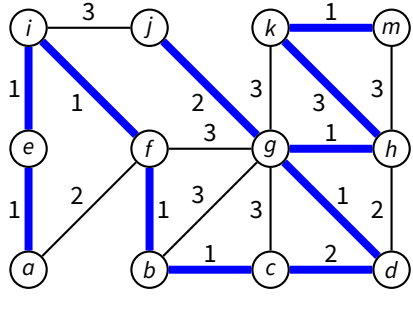

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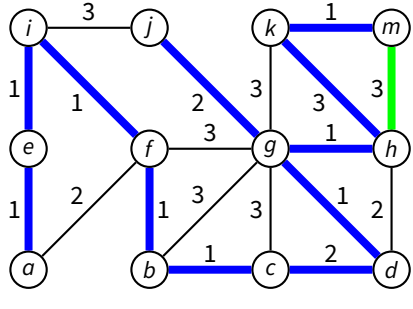

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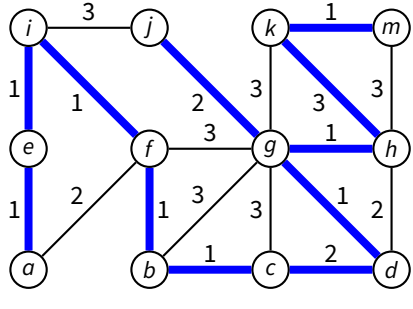

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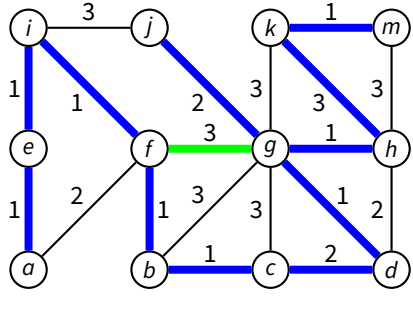

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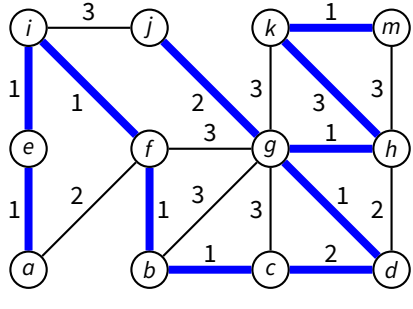

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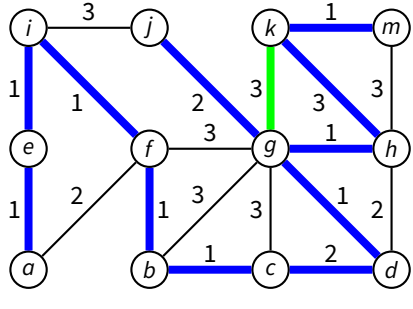

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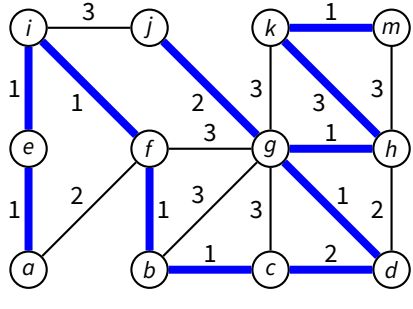

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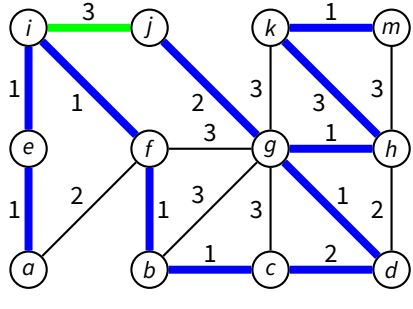

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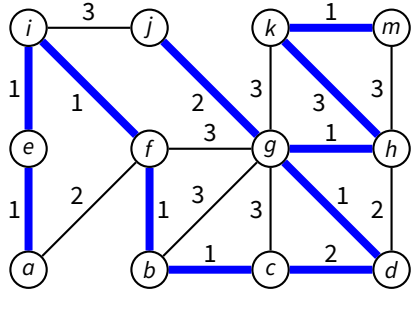

Complexity of MST-KRUSKAL

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Complexity of MST-KRUSKAL

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MST-KRUSKAL(G, w)
1  A = ∅
2  for each vertex v ∈ V(G)
3      MAKE-SET(v)                    // disjoint-set data structure
4  sort E in non-decreasing order by weight w
5  for each edge (u, v) ∈ E, taken in non-decreasing order by w
6      if FIND-SET(u) ≠ FIND-SET(v)
7          A = A ∪ {(u, v)}
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- $|V|$ times **MAKE-SET** (loop of line 2–3)

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  - T, tree being built
  - $W[v]$ best known cost/weight of connecting v with T
  - $P[v]$, node $u \in T$ such that the edge (u, v) is the least-cost edge connecting v with T

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MST-PRIM(G, u, w)
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 5  W[u] = 0
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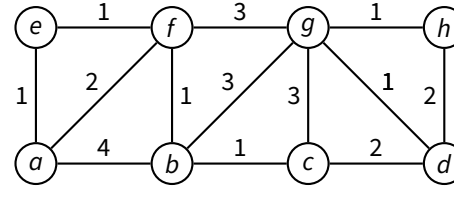

v	W	P	T
a	0		
b	∞		
c	∞		
d	∞		
e	∞		
f	∞		
g	∞		
h	∞		

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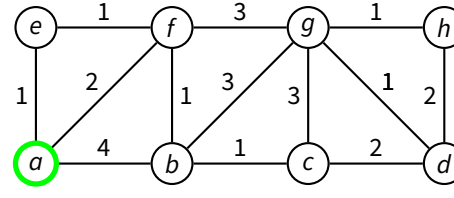

v	W	P	T
a	0		✓
b	∞		
c	∞		
d	∞		
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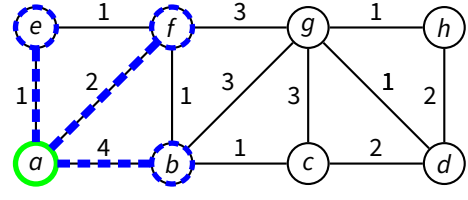

v	W	P	T
a	0		✓
b	4	a	
c	∞		
d	∞		
e	1	a	
f	2	a	
g	∞		
h	∞		

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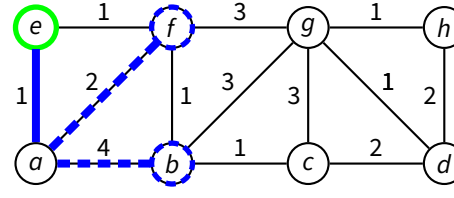

v	W	P	T
a	0		✓
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c	∞		
d	∞		
e	1	a	✓
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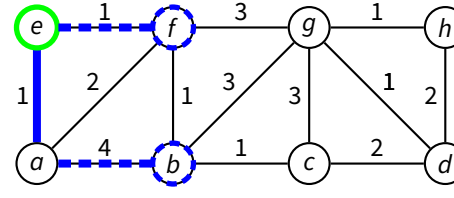

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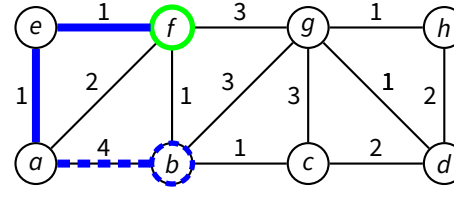

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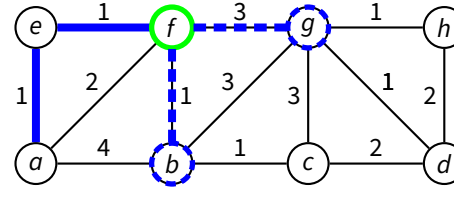

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a	0		✓
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d	∞		
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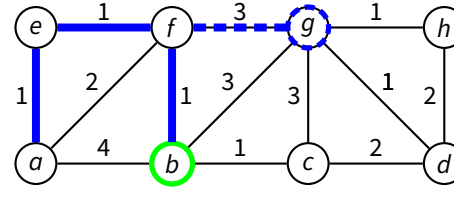

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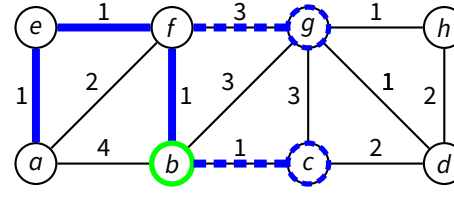

v	W	P	T
a	0		✓
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c	1	b	
d	∞		
e	1	a	✓
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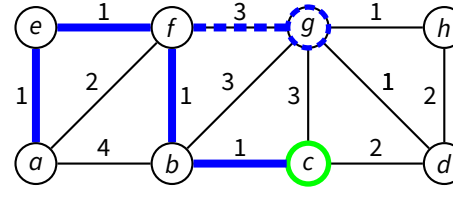

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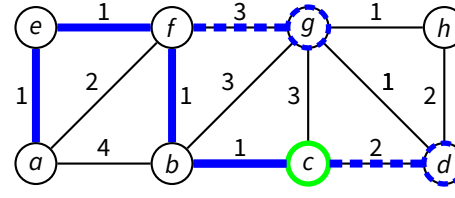

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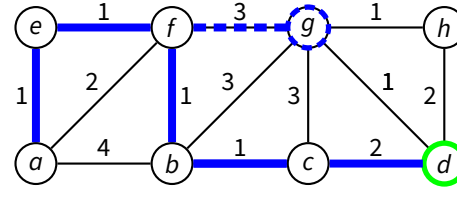

v	W	P	T
a	0		✓
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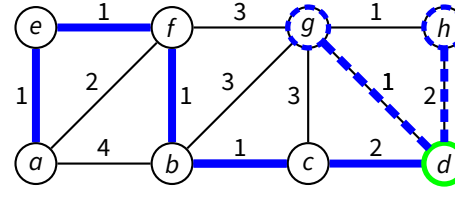

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a	0		✓
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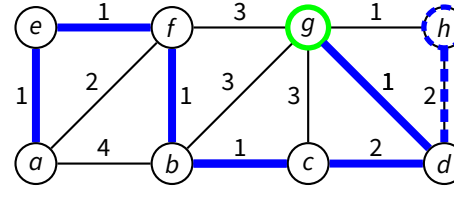

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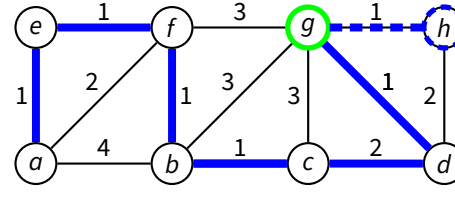

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a	0		✓
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f	1	e	✓
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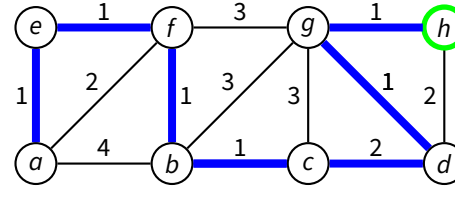

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a	0		✓
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c	1	b	✓
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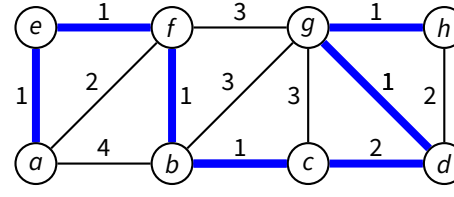

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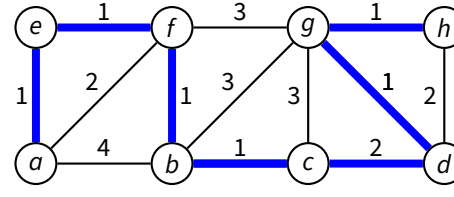
e	1	a	✓
f	1	e	✓
g	1	d	✓
h	1	g	✓

Prim's Algorithm

```
MST-PRIM(G, u, w)
 1  T = (∅, ∅)
 2  for each vertex v ∈ V(G)
 3       W[v] = ∞
 4       P[v] = NIL
 5  W[u] = 0
 6  while V(T) ≠ V(G)
 7       find u ∉ V(T) such that W[u] is minimal
 8       T = T ∪ {u}  // add u to T
 9       for all v ∈ Adj(u) \ V(T)
10            if w(u, v) < W[v]
11                 W[v] = w(u, v)
12                 P[v] = u
```



v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	✓
d	2	c	✓
e	1	a	✓
f	1	e	✓
g	1	d	✓
h	1	g	✓