Elementary Data Structures and Hash Tables

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Outline

- Common concepts and notation
- Stacks
- Queues
- Linked lists
- Trees
- Direct-access tables
- Hash tables

Concepts

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- A data structure stores ***data*** and possibly ***meta-data***
  - e.g., a *heap* needs an array A to store the keys, plus a variable $A.\mathit{heap\text{-}size}$ to remember how many elements are in the heap

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  - **STACK-EMPTY**(S) returns TRUE if and only if S is empty
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- *Implementation*
  - using an array
  - using a linked list
  - ...

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STACK-EMPTY(S)
if S.top == 0
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```
PUSH(S,x)
S.top = S.top + 1
S[S.top] = x
```

```
POP(S)
if STACK-EMPTY(S)
        error "underflow"
else S.top = S.top − 1
        return S[S.top + 1]
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  - **ENQUEUE**(Q, x) adds element x at the back of queue Q
  - **DEQUEUE**(Q) extracts the element at the head of queue Q
- *Implementation*
  - Q is an array of fixed length $Q.length$
    - i.e., Q holds at most $Q.length$ elements
    - enqueueing more than Q elements causes an "overflow" error
  - $Q.head$ is the position of the "head" of the queue
  - $Q.tail$ is the first empty position at the tail of the queue

Enqueue

```
ENQUEUE(Q,x)
if Q.queue-full
        error "overflow"
else Q[Q.tail] = x
        if Q.tail < Q.length
                Q.tail = Q.tail + 1
        else Q.tail = 1
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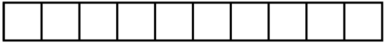

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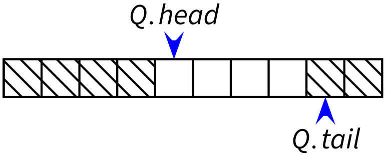

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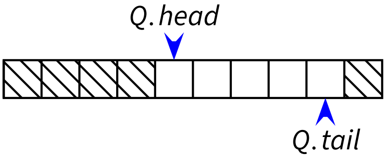

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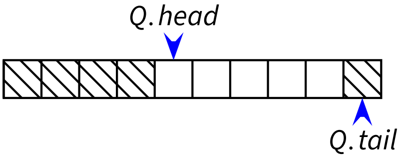

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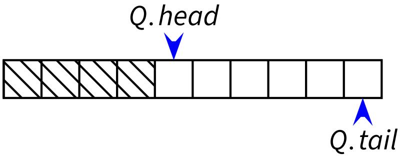

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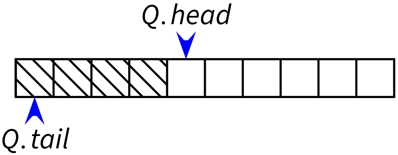

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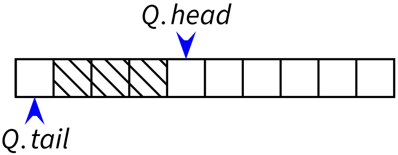

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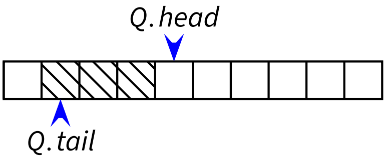

Dequeue

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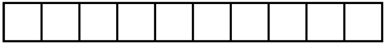

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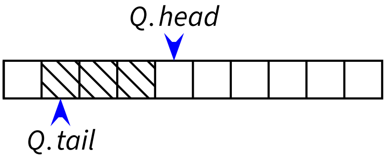

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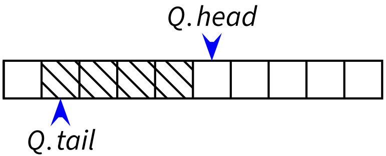

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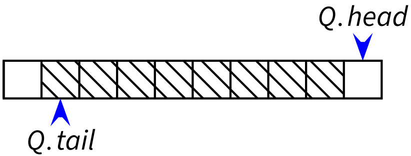

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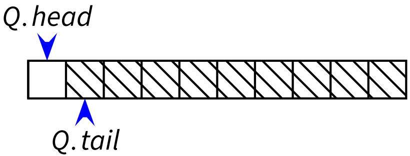

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- *Implementation*
  - a *doubly-linked* list
  - each element x has two “links” $x.prev$ and $x.next$ to the previous and next elements, respectively
  - each element x holds a key $x.key$
  - it is convenient to have a dummy “sentinel” element $L.nil$

Linked List With a “Sentinel”

LIST-INIT(L)

```
L.nil.prev = L.nil
L.nil.next = L.nil
```

LIST-INSERT(L, x)

```
x.next = L.nil.next
L.nil.next.prev = x
L.nil.next = x
x.prev = L.nil
```

LIST-SEARCH(L, k)

```
x = L.nil.next
while x ≠ L.nil ∧ x.key ≠ k
    x = x.next
return x
```

Complexity

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STACK-EMPTY	

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LIST-INSERT	$O(1)$
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LIST-SEARCH	$\Theta(n)$

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Direct-Address-Insert(T, k)

```
1  T[k] = TRUE
```

Direct-Address-Delete(T, k)

```
1  T[k] = FALSE
```

Direct-Address-Search(T, k)

```
1  return T[k]
```

Direct-Address Table (2)

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- The ***space complexity*** is $\Theta(|U|)$
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- *Can we have the benefits of a direct-address table but with a table of reasonable size?*

- *Idea*
  - use a table T with $|T| \ll |U|$
  - map each key $k \in U$ to a position in T, using a ***hash function***

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HASH-INSERT(T, k)
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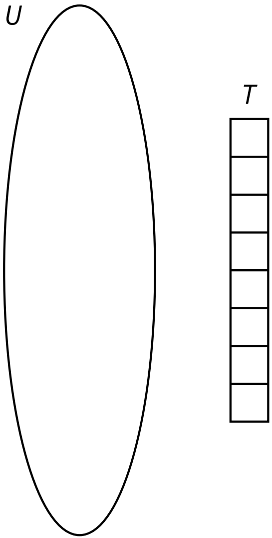
Are these algorithms correct? No!

What if two distinct keys $k_1 \neq k_2$ *collide?* (I.e., $h(k_1) = h(k_2)$)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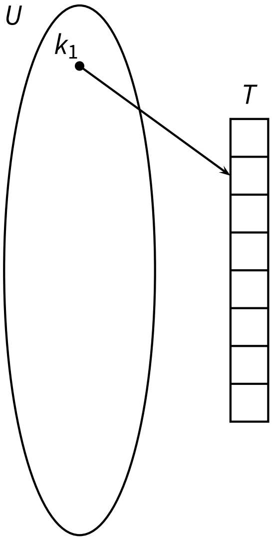
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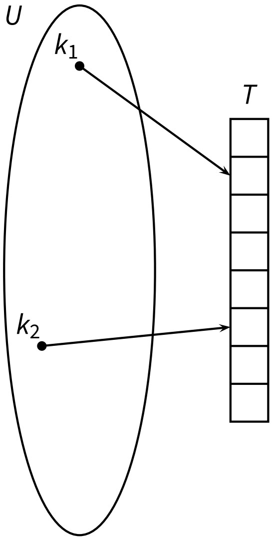

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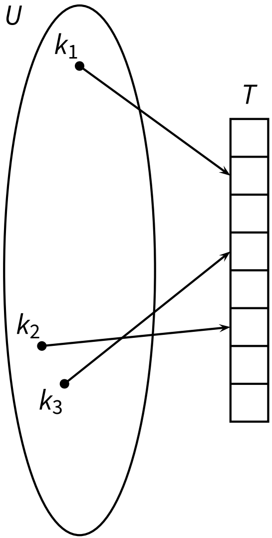

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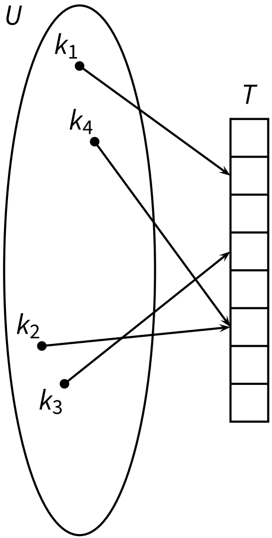

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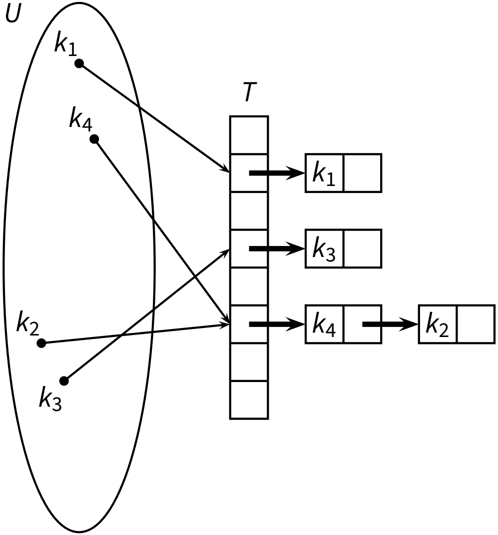

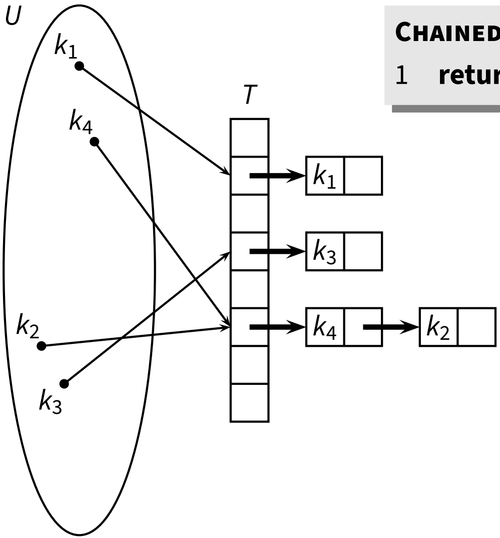

CHAINED-HASH-INSERT(T, k)

1 **return LIST-INSERT**$(T[h(k)], k)$

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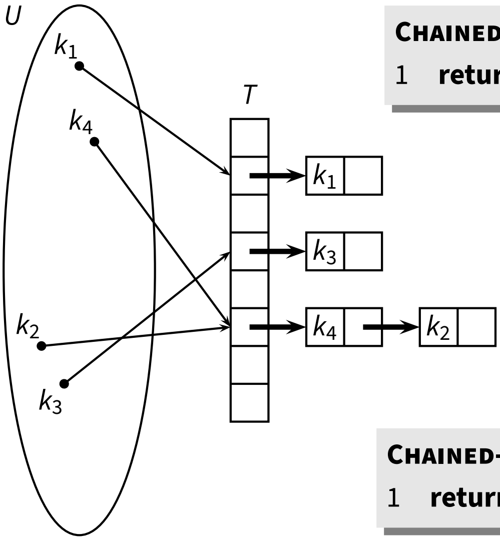

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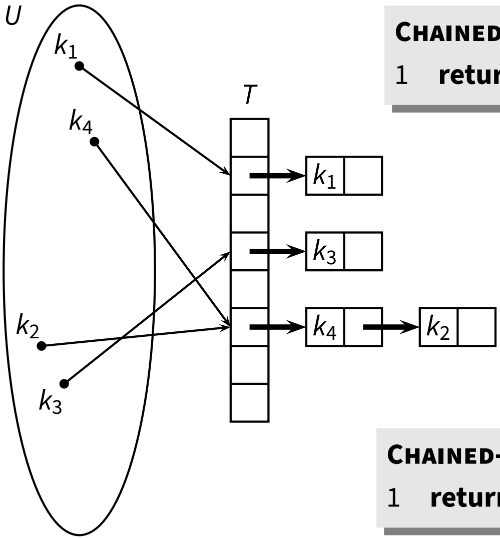

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load factor

$$\alpha = \frac{n}{|T|}$$

Chained-Hash-Search(T, k)

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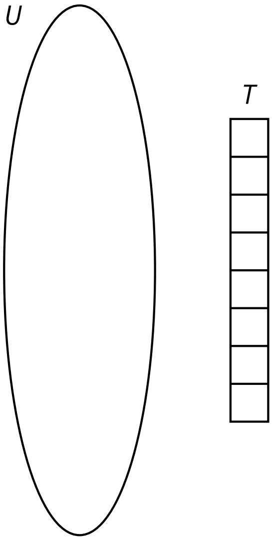
- We further assume that $h(k)$ can be computed in $O(1)$ time

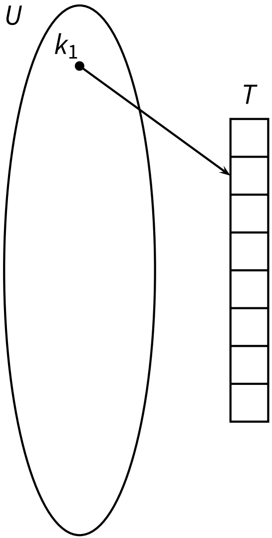
- Therefore, the complexity of **CHAINED-HASH-SEARCH** is

$$\boxed{\Theta(1 + \alpha)}$$

Open-Address Hash Table



Open-Address Hash Table



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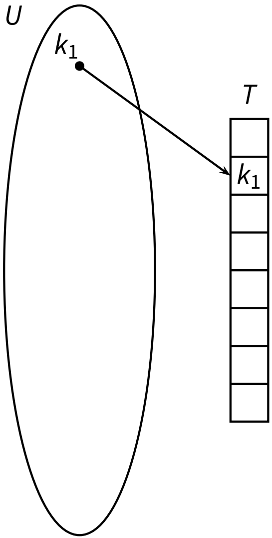
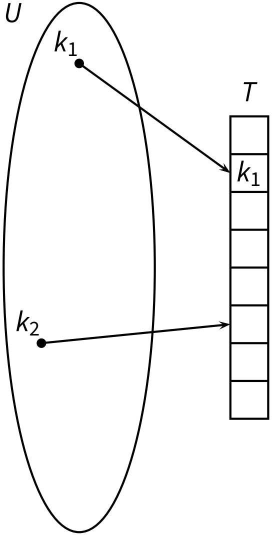
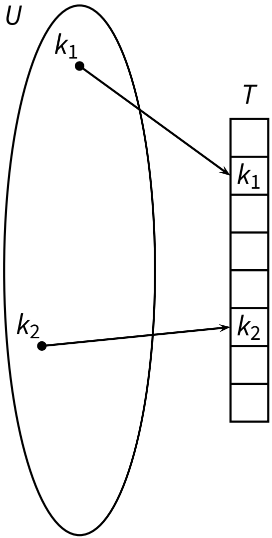


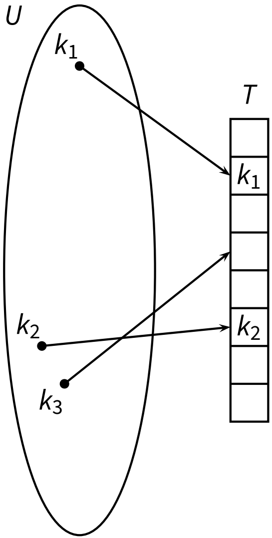
Open-Address Hash Table



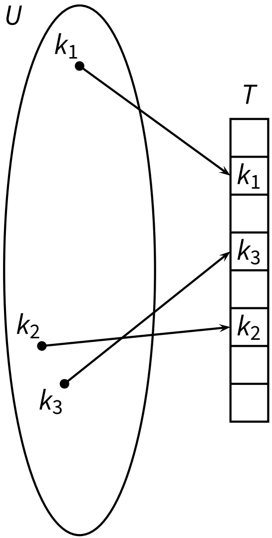
Open-Address Hash Table



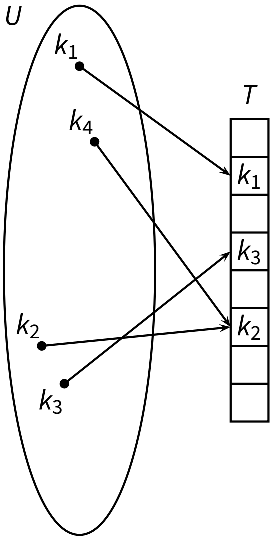
Open-Address Hash Table



Open-Address Hash Table



Open-Address Hash Table



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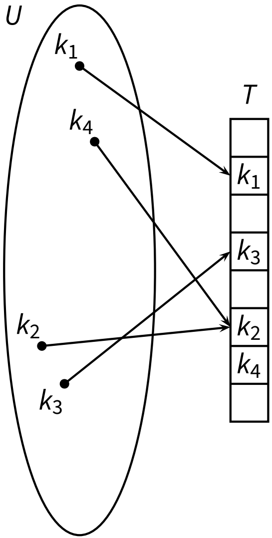

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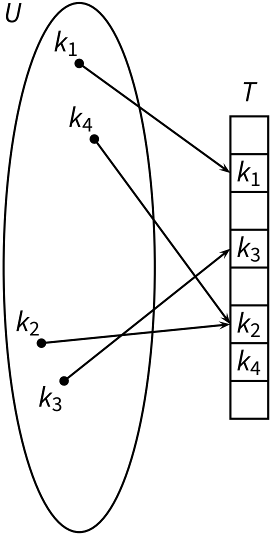

HASH-INSERT(T, k)

```
j = h(k)
for i = 1 to T.length
    if T[j] == NIL
        T[j] = k
        return j
    elseif j < T.length
        j = j + 1
    else j = 1
error "overflow"
```

Open-Addressing (2)

- *Idea:* instead of using linked lists, we can store all the elements in the table
  - this implies $\alpha \leq 1$

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  - this implies $\alpha \leq 1$
- When a collision occurs, we simply find another free cell in T
- A sequential “probe” may not be optimal
  - can you figure out why?

```
HASH-INSERT(T, k)
for i = 1 to T.length
j = h(k, i)
    if T[j] == NIL
        T[j] = k
        return j
error "overflow"
```

Open-Addressing (3)

```
HASH-INSERT(T, k)
for i = 1 to T.length
    j = h(k, i)
    if T[j] == NIL
        T[j] = k
        return j
error "overflow"
```

- Notice that $h(k, \cdot)$ must be a ***permutation***
  - i.e., $h(k,1), h(k,2), \ldots, h(k,|T|)$ must cover the entire table T