Divide-and-Conquer Algorithms

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Outline

- Merging (or set union)
- Searching
- Sorting
- Multiplying
- Computing the *median*

Merging (Set Union)

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- **Input:** sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

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- **Example:**

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  $B = \langle 51, 21, 14, 15, 27, 31, 2 \rangle$

  $X =$

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  - iterate through every position i, first through A, and then B
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MERGESIMPLE(A, B)
1  for i = 1 to length(A)
2       if not FIND(A[1 . . i − 1], A[i])
3            output A[i]
4  for i = 1 to length(B)
5       if not FIND(A, B[i]) and not FIND(B[1 . . i − 1], B[i])
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$$\text{let } n = length(A) + length(B)$$

$$T(n) = \sum_{i=1}^{length(A)} T_{\textbf{FIND}}(i) + \sum_{i=1}^{length(B)} \left(T_{\textbf{FIND}}(i) + T_{\textbf{FIND}}(length(A))\right)$$

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Searching

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  Output: TRUE if A contains *key*, or FALSE otherwise

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FIND(A, begin, end, key)
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FINDINLIST(A, key)
1  item = first(A)
2  while item ≠ last(A)
3        if value(item) == key
4              return TRUE
5        item = next(item)
6  return FALSE
```

Searching in a List

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  Output: TRUE if A contains *key*, or FALSE otherwise

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Complexity of MERGESIMPLE

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$$T(n) = \sum_{i=1}^{n} T_{\textsc{Find}}(i)$$

Complexity of MergeSimple

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Complexity of MergeSimple

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BINARYSEARCH(A, key)
 1  first = 1
 2  last = length(A)
 3  while first ≤ last
 4        middle = ⌈(first + last)/2⌉
 5        if A[middle] == key
 6              return TRUE
 7        elseif first = last
 8              return FALSE
 9        elseif A[middle] > key
10              last = middle − 1
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12  return FALSE
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Binary Search

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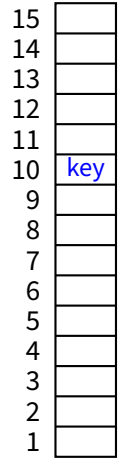

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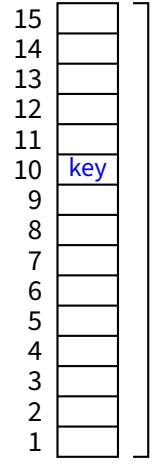

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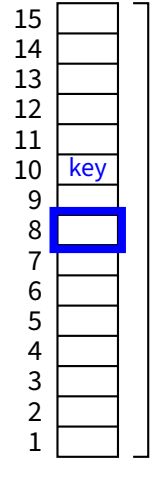

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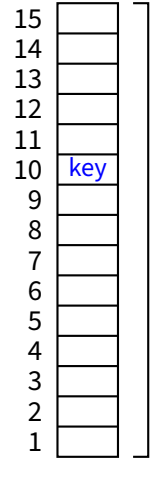

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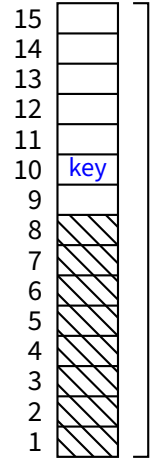

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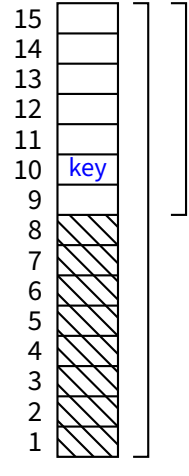

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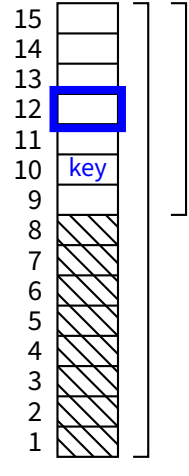

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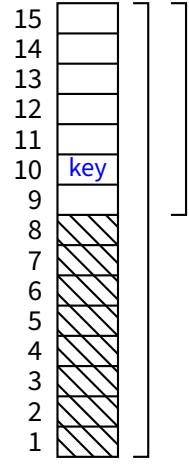

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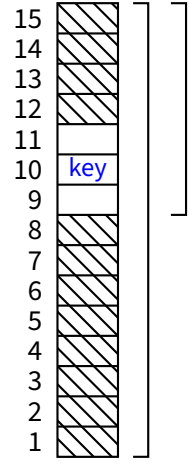

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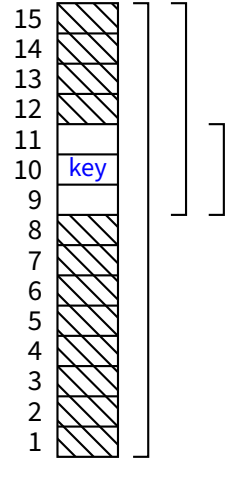

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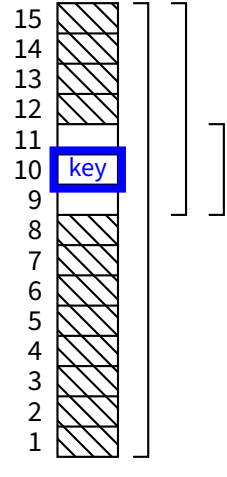

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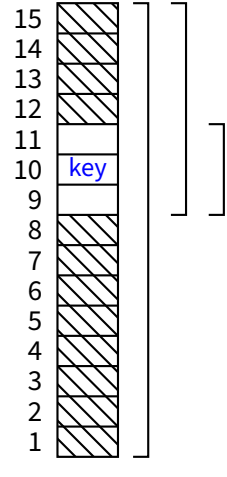

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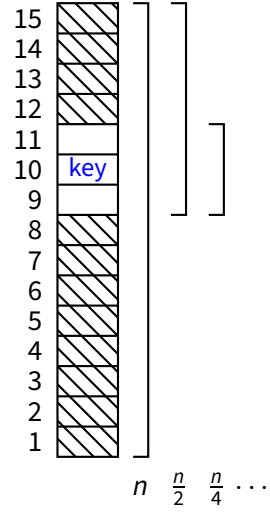

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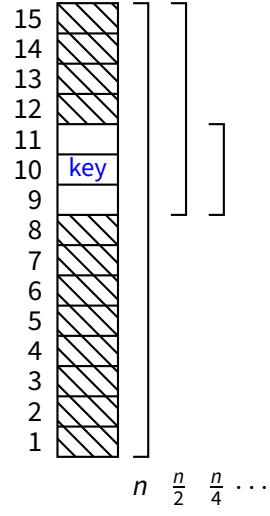

$$T(n) = O(\log n)$$

Merging Sorted Sequences

- A slightly different problem:

  Input: two *sorted* sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$, where $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_m$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

  - every element of A appears once in X
  - every element of B appears once in X
  - every element of X appears in A or in B or in both

A Better Merge Algorithm

```
MERGESIMPLE2(A, B)
1  for i = 1 to length(A)
2      if not BINARYSEARCH(A[1..i − 1], A[i])
3          output A[i]
4  for i = 1 to length(B)
5      if not BINARYSEARCH(A, B[i])
6      and not BINARYSEARCH(B[1..i − 1], B[i])
7          output B[i]
```

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$$T(n) = \sum_{i=1}^{n} O(\log i) =$$

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```

$$T(n) = \sum_{i=1}^{n} O(\log i) = O(n \log n)$$

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MERGESIMPLE2(A, B)
1  for i = 1 to length(A)
2      if not BINARYSEARCH(A[1..i − 1], A[i])
3          output A[i]
4  for i = 1 to length(B)
5      if not BINARYSEARCH(A, B[i])
6      and not BINARYSEARCH(B[1..i − 1], B[i])
7          output B[i]
```

$$T(n) = \sum_{i=1}^{n} O(\log i) = O(n \log n)$$

Better than $O(n^2)$, but can we do even better than $O(n \log n)$?

An Even Better Merge Algorithm

- *Intuition:* A and B are *sorted*
  e.g.
  $A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$
  $B = \langle 1, 5, 6, 7, 34, 35, 40, 41, 43 \rangle$

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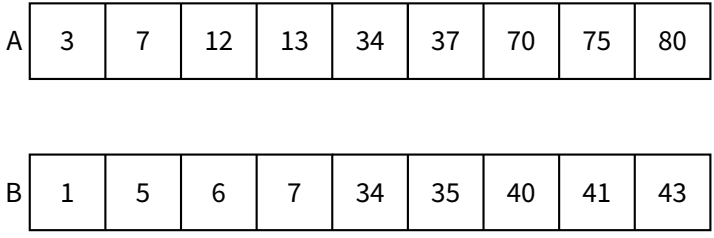
  so just like in **BinarySearch** I can avoid looking for an element x if the *first* element I see is $y > x$

- High-level algorithm strategy
  - step through every position i of A and every position j of B
  - output a_i and advance i if $a_i \leq b_j$ or if j is beyond the end of B
  - output b_j and advance j if $a_i \geq b_j$ or if i is beyond the end of A

MERGE Algorithm



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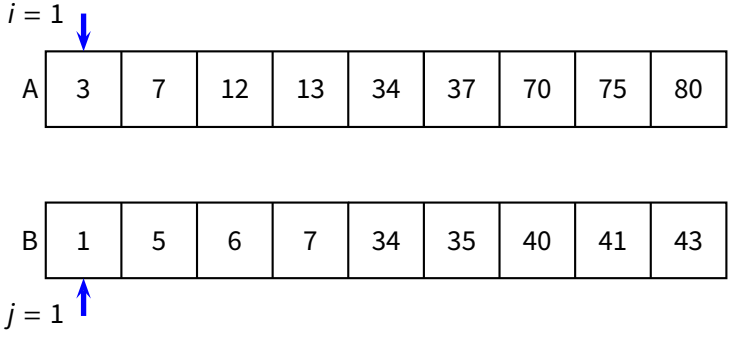
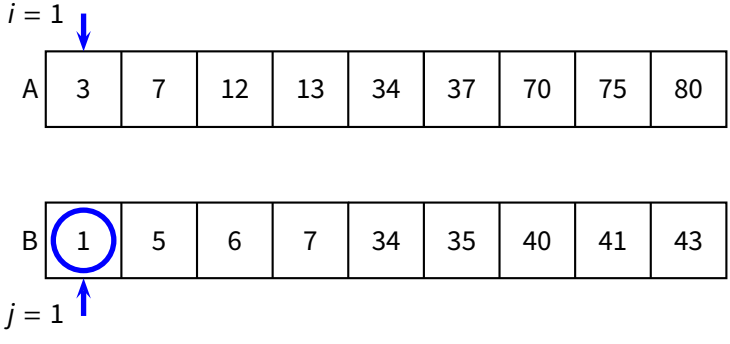


Output:

MERGE Algorithm



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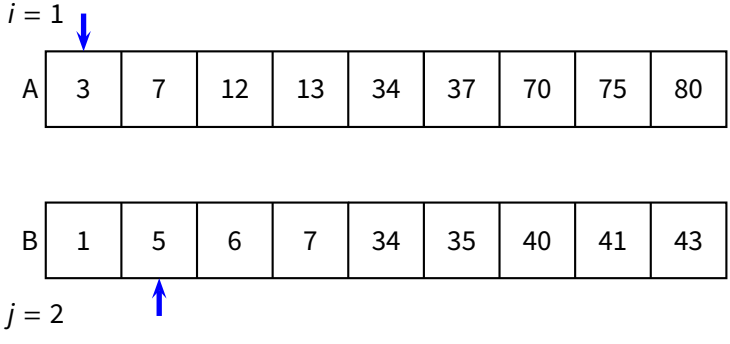
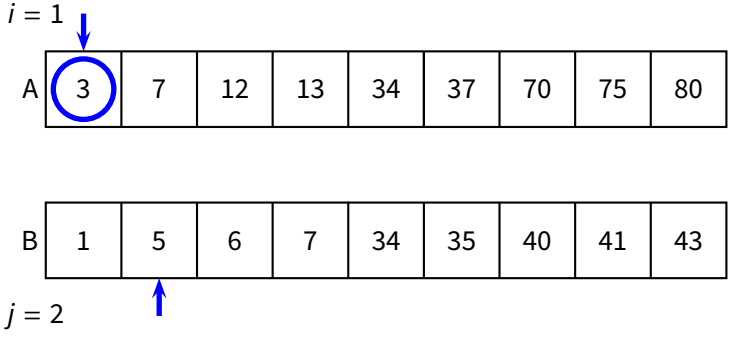


Output: 1



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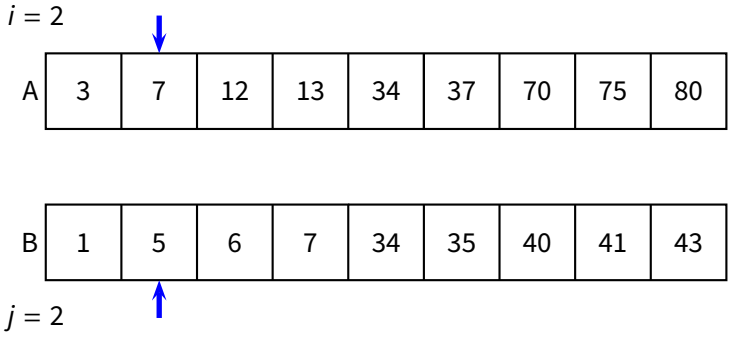

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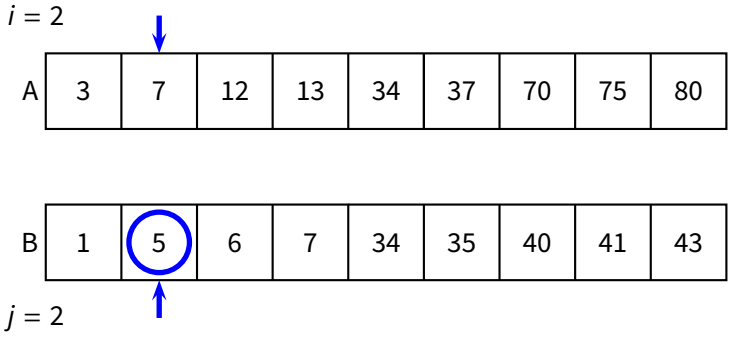

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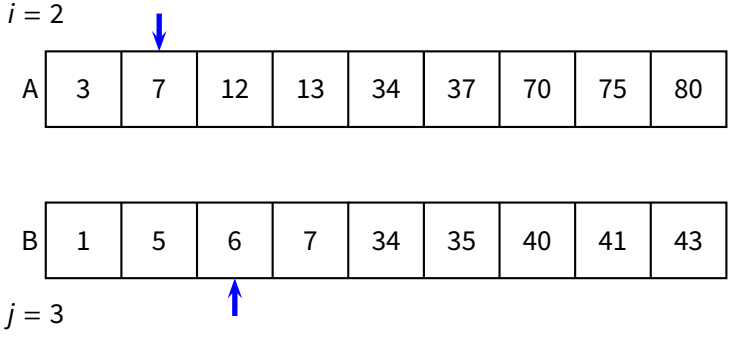

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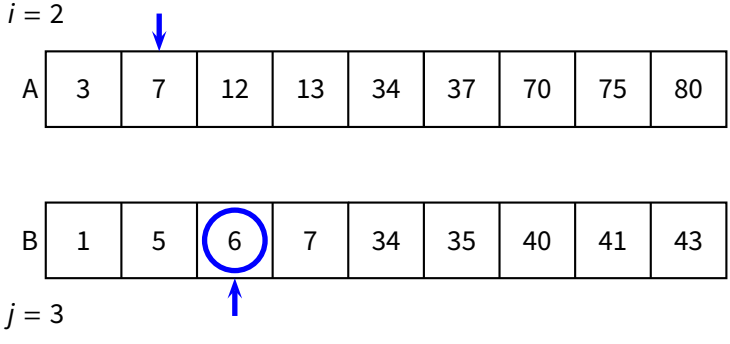

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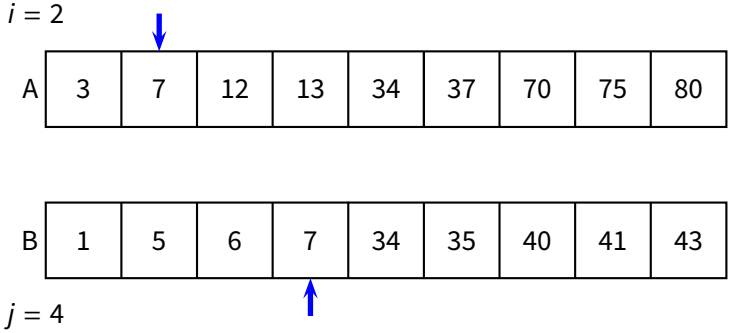

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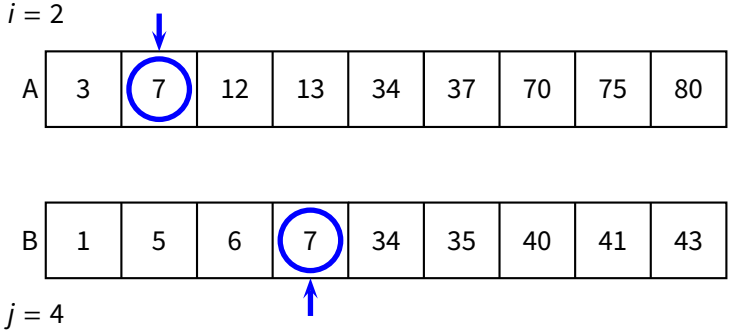

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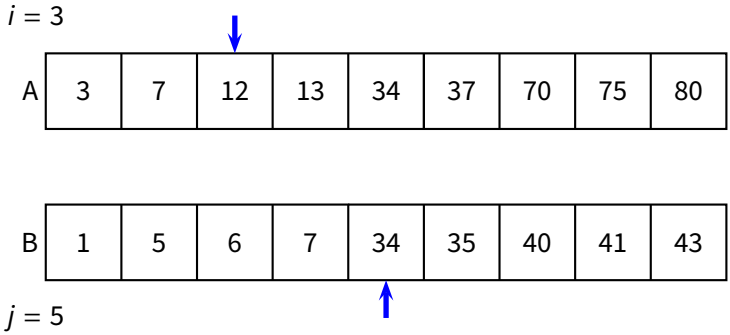

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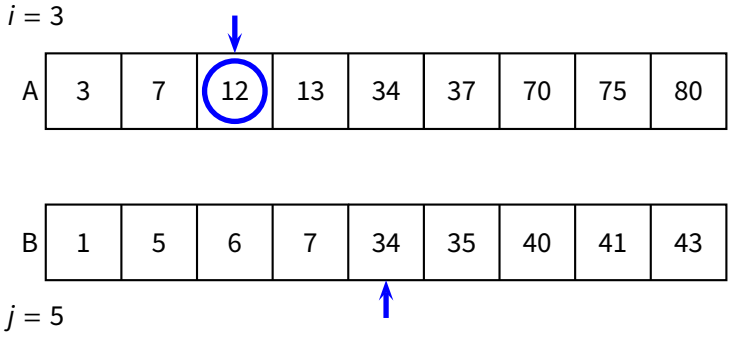

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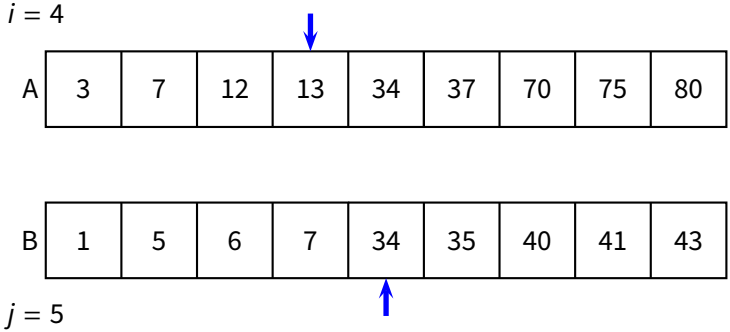

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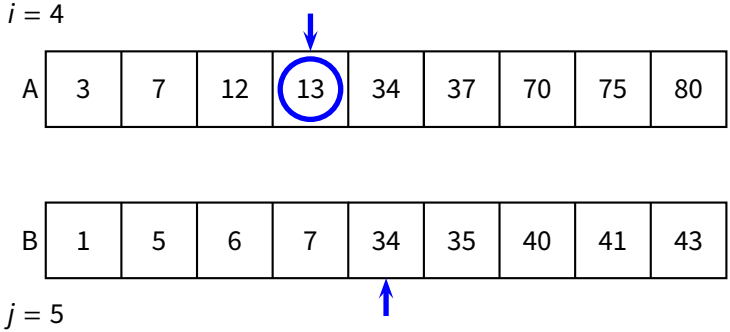

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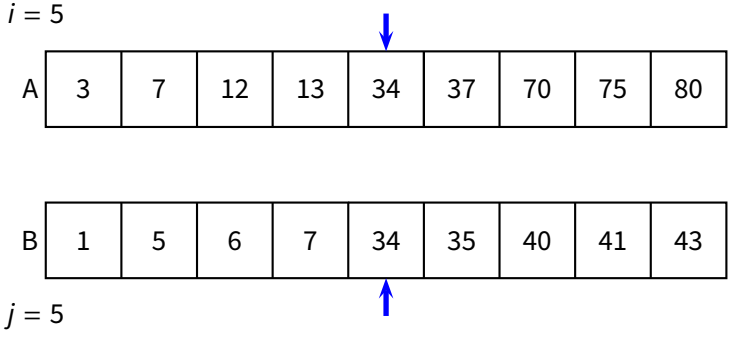

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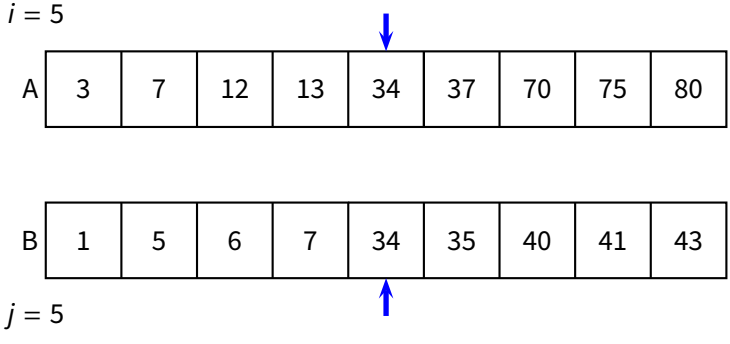

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MERGE Algorithm (2)

```
MERGE(A, B)
 1  i, j = 1
 2  X = ∅
 3  while i ≤ length(A) or j ≤ length(B)
 4        if i > length(A)
 5              X = X ∘ B[j]          // appends B[j] to X
 6              j = j + 1
 7        elseif j > length(B)
 8              X = X ∘ A[i]
 9              i = i + 1
10        elseif A[i] < B[j]
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- This algorithm is incorrect! (Exercise: fix it)

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- Can we do better? No!
  - we have to output $n = length(A) + length(B)$ elements

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  - merges two *sorted* sequences
  - *produces a sorted sequence*

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  - this suggests a recursive algorithm

Merge Sort

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```
MERGESORT(A)
1  if length(A) == 1
2        return A
3  m = ⌊length(A)/2⌋
4  A_L = MERGESORT(A[1 . . m])
5  A_R = MERGESORT(A[m + 1 . . length(A)])
6  return MERGE(A_L, A_R)
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$$\boxed{T(n) = O(n \log n)}$$

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- *General strategy:* given a problem P on input data A
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- Complexity analysis

$$T(n) = T_{\text{divide}} + \sum_{i=1}^{k} T(|A_i|) + T_{\text{combine}}$$

we might analyze this formula another time…

A Divide-and-Conquer Merge

```
MERGER(A, B)
1   if length(A) == 0
2         return B
3   if length(B) == 0
4         return A
5   if A[1] < B[1]
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- Can we do better? No! (We knew that already)

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$x =$ | X_L | X_R | and $y =$ | Y_L | Y_R |

Divide-and-Conquer Multiplication

- Going back to multiplication…

  $x =$ $\boxed{X_L}\boxed{X_R}$ and $y =$ $\boxed{Y_L}\boxed{Y_R}$

  which means $x = 2^{\ell/2}x_L + x_R$ and $y = 2^{\ell/2}y_L + y_R$, so…

$$\begin{aligned} xy &= (2^{\ell/2}x_L + x_R)(2^{\ell/2}y_L + y_R) \\ &= 2^{\ell}x_Ly_L + 2^{\ell/2}(x_Ly_R + x_Ry_L) + x_Ry_R \end{aligned}$$

  we reduced the problem of multiplying two numbers of ℓ bits into the problem of multiplying *four* numbers of $\ell/2$ bits…

Divide-and-Conquer Multiplication

- Going back to multiplication…

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but notice that $x_L y_R + x_R y_L = (x_L + x_R)(y_R + y_L) - x_L y_L - x_R y_R$, so

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which, as we will see, leads to a much better complexity

$$T(\ell) = O(\ell^{\log_2 3}) = O(\ell^{1.59})$$

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    the 6th smallest element of A—a.k.a. $select(A, 6)$—is 8

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It is the 2nd smallest value of A_R

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We use $select(A, k)$ to denote the k-smallest element of A

$$select(A, k) = \begin{cases} select(A_L, k) & \text{if } k \leq |A_L| \\ v & \text{if } |A_L| < k \leq |A_L| + |A_v| \\ select(A_R, k - |A_L| - |A_v|) & \text{if } k > |A_L| + |A_v| \end{cases}$$

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- We pick *a random element of A*

Selection Algorithm

```
SELECTION(A, k)
 1  v = A[random(1 ... |A|)]
 2  A_L, A_v, A_R = ∅
 3  for i = 1 to |A|
 4        if A[i] < v
 5              A_L = A_L ∪ A[i]
 6        elseif A[i] == v
 7              A_v = A_v ∪ A[i]
 8        else A_R = A_R ∪ A[i]
 9  if k ≤ |A_L|
10        return SELECTION(A_L, k)
11  elseif k > |A_L| + |A_v|
12        return SELECTION(A_R, k − |A_L| − |A_v|)
13  else return v
```