Basic Elements of Complexity Theory

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Outline

- Basic complexity classes
- Polynomial reductions
- NP-completeness

Polynomial Time

- A ***polynomial-time algorithm*** is one whose worst-case running time $T(n)$, on an input of size n bits, is $O(n^k)$ for some *constant* k

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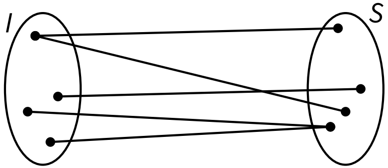
Problems

- A ***problem*** Q is a binary relation between a set I of ***instances*** and a set S of ***solutions***



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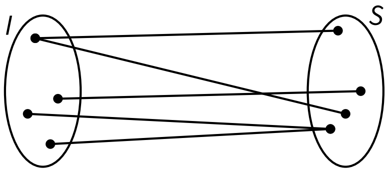

- A ***concrete problem*** Q is one where I and S are the set of binary strings $\{0, 1\}^*$
  - for all practical purposes, instances and solutions can be ***encoded*** as binary strings (i.e., mapped into $\{0, 1\}^*$)
  - we consider only sensible encodings…

Decision Problems

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Example:

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Primality Testing

Decision vs. Optimization: Example

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- Shortest path in a graph

$$G = (V = \{a, b, c, \ldots\}, E = \{(a, c), \ldots\}), a, z \longrightarrow a, c, \ldots, z$$

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- *Shortest path as a* ***decision problem***

$$G = (V = \{a, b, c, \dots\}, E = \{(a, c), \dots\}), a, z, 10 \longrightarrow 1$$

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  - *input:* a graph G, a start vertex (a), an end vertex (z), and a path length (10)
  - *output:* 1 if there is a path of (at most) the given length

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- An optimization problem is ***not much harder*** than the corresponding decision problem
  - having a solution to the decision problem does not give an immediate solution to the optimization problem
  - but we can typically use the decision problem as a subroutine in some kind of (binary) search to solve the corresponding optimization problem

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  - parsing a Java program
  - …

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  - *Input:* A graph $G = (V, E)$ and a number K
  - *Output:* 1, if there is set S of at most k vertices such that for every edge $e = (u, v) \in E$, $u \in S$ or $v \in S$ (or both); 0 otherwise

Verifying is Easy

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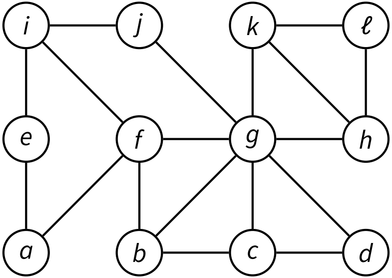

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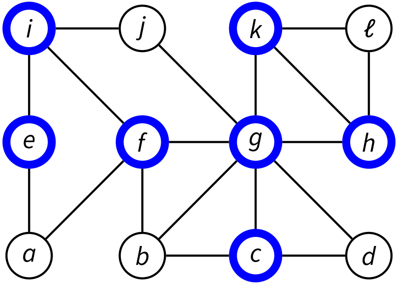

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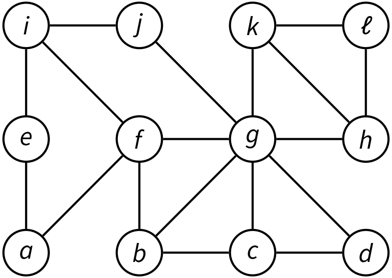
$K = 6$?



Polynomial-Time Verification

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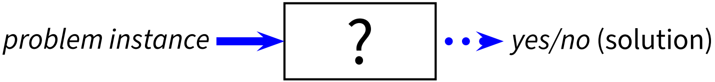

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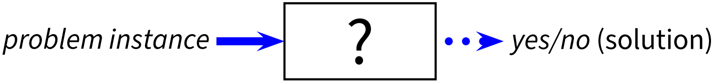

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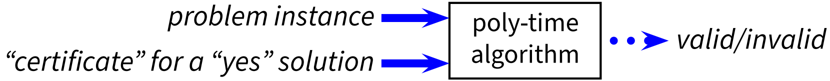

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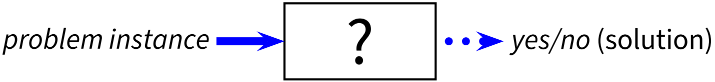

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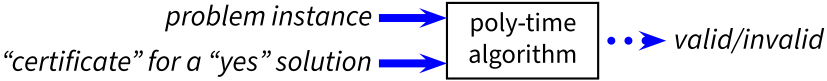

- Examples
  - longest path (decision variant)
  - knapsack (decision variant)

The Complexity Class NP

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  - there is a polynomial-time algorithm A
  - for each instance $x \in I$ that has a "yes" solution ($Q(x) = 1$)
  - there is a ***certificate*** y of polynomial-size $|y| = O(|x|^c)$, for some constant c
  - such that $A(x, y) = 1$

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- *polynomial-time solvable* $\Longrightarrow$ *polynomial-time verifiable*

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Finding a solution to a problem is believed to be inherently more difficult than verifying a given solution (or a proof of a solution)

. . . but nobody has been able to prove that this is the case!

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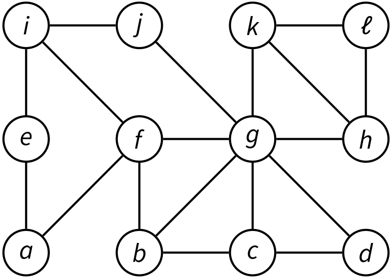

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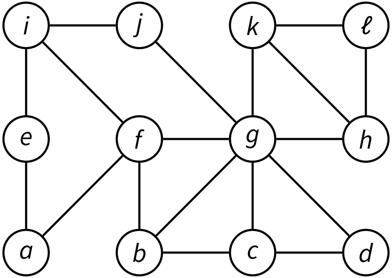

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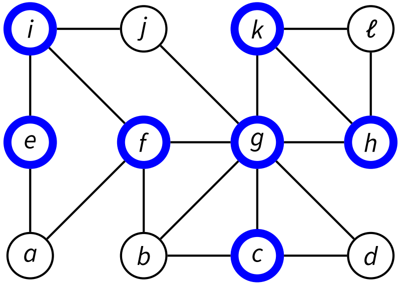

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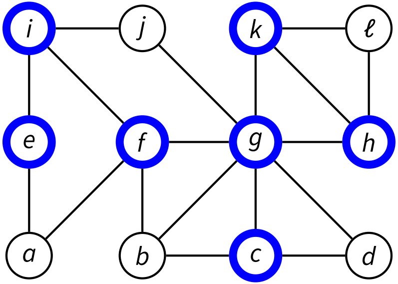

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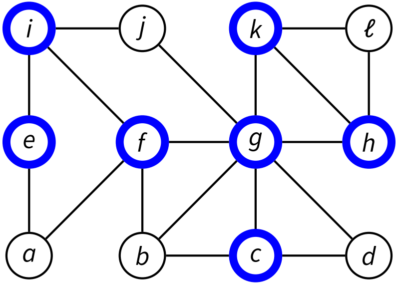

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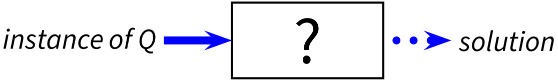

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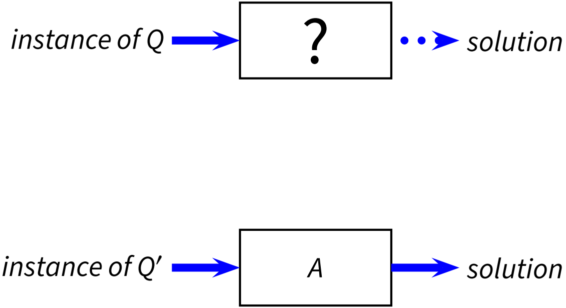

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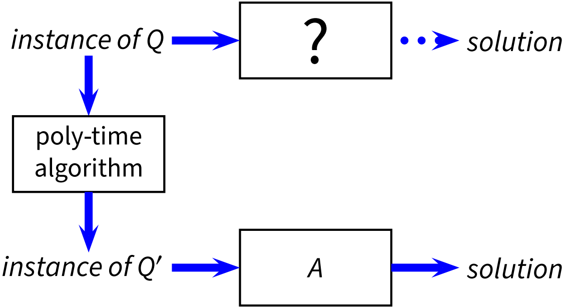

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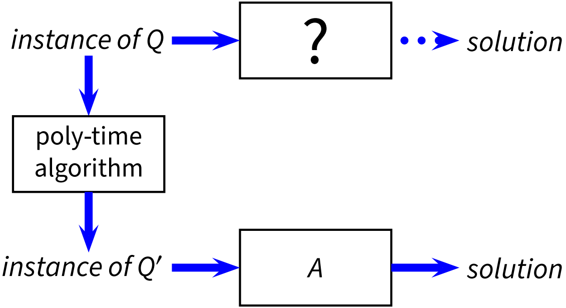

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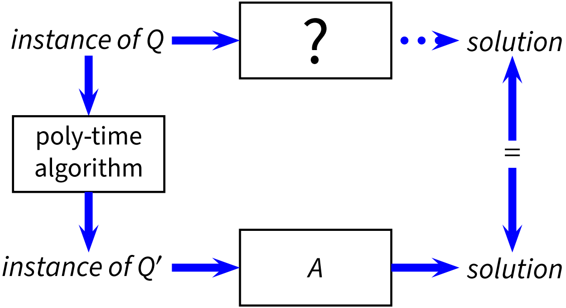
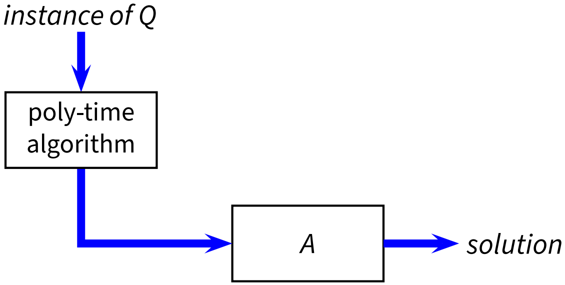


  - an instance q of Q is transformed into an instance q' of Q' through a polynomial-time algorithm
  - the solution to q is 1 if and only if the solution to q' is 1

Reduction (2)

- Solution by polynomial-time reductions to a solvable problem



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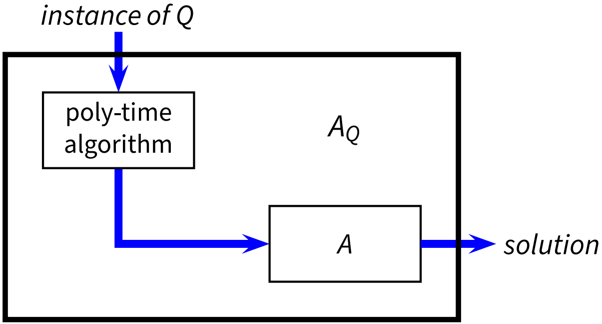

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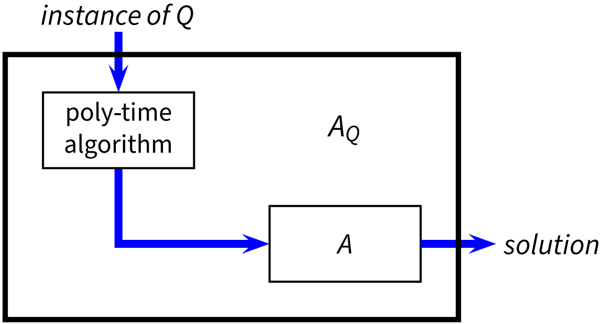

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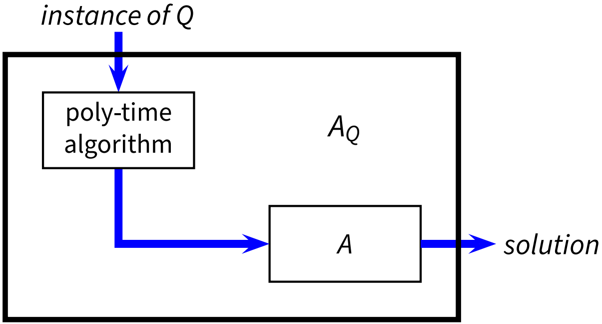

  - if A is polynomial-time, then of A_Q is also polynomial time
  - therefore if $Q' \in P$, then $Q \in P$

Example: 2-CNF-SAT

2-CNF-SAT problem

Input:

- f is a Boolean formula of n (Boolean) variables $x_1, x_2, \ldots, x_n$
- f is in *conjunctive normal form (CNF)*, so $f = C_1 \wedge C_2 \wedge \cdots \wedge C_k$
- every *clause* C_i of f contains exactly *two* literals (a variable or its negation)

Output: 1 iff f is satisfiable

- there is an assignment of variables that satisfies f

Example:

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2)$$

2-CNF-SAT to Implicative Form

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- Consider each clause C_i

$$(a \vee b) \equiv (\neg a \Rightarrow b) \equiv (\neg b \Rightarrow a)$$

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$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3)$$

is equivalent to

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2-CNF-SAT to Graph Reachability

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2)$$

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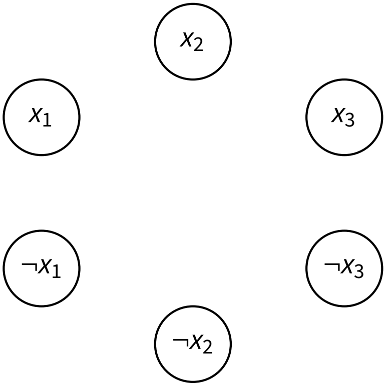

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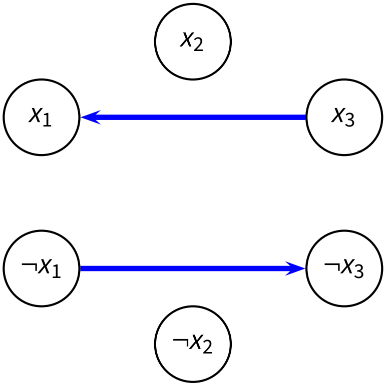

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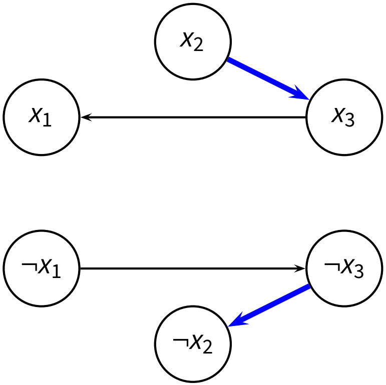

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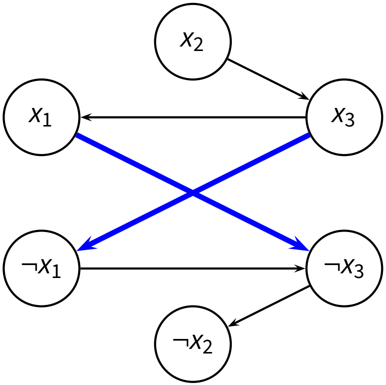

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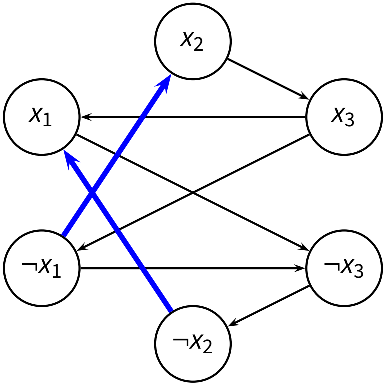

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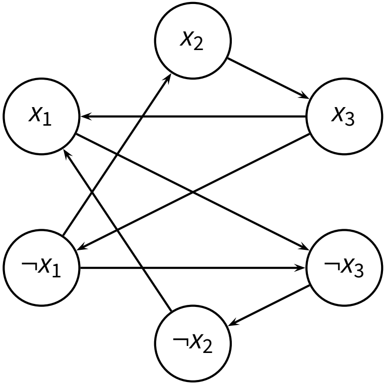

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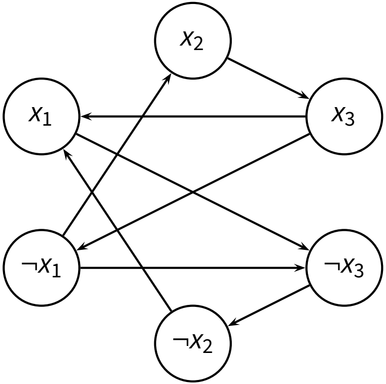

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depth-first search

Reduction of 2-CNF-SAT

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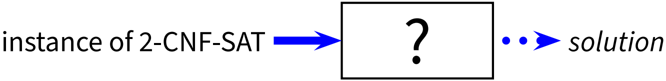
- 2-CNF-SAT $\in P$



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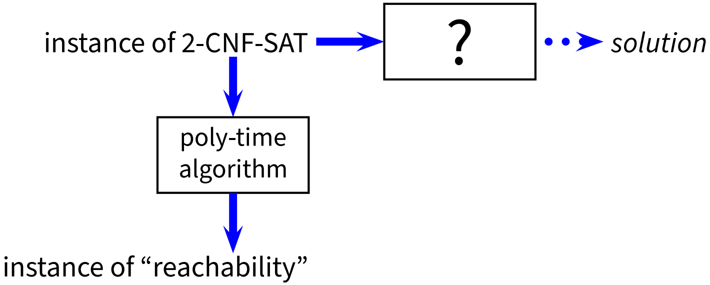
instance of 2-CNF-SAT
?
solution
poly-time
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instance of “reachability”

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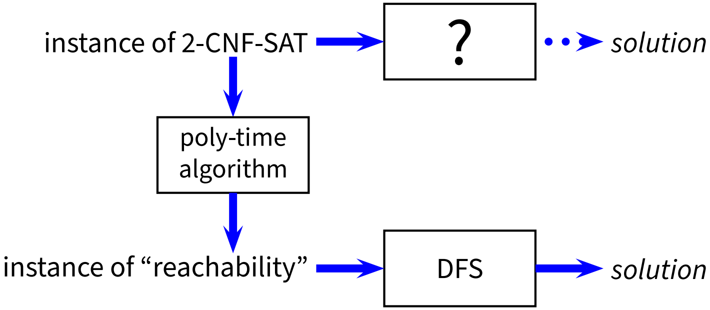
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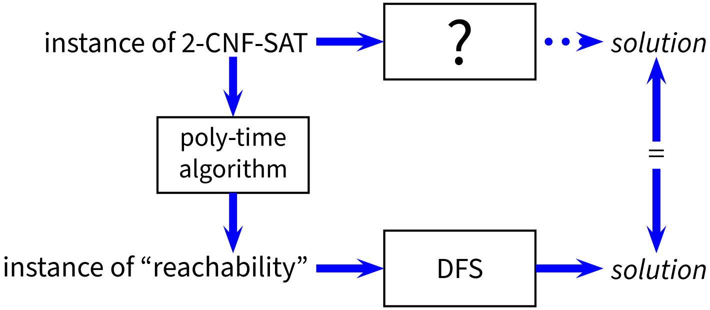
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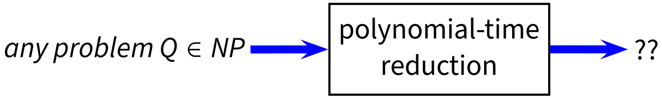
- If Q' is NP-hard and *polynomial-time reducible* to Q'', then Q'' is NP-hard
- If Q' is NP-hard and *polynomial-time solvable*, then $\mathrm{P} = \mathrm{NP}$
  - most researchers believe that there is no such Q'

The First NP-Complete Problem

- Is there any NP-complete problem?



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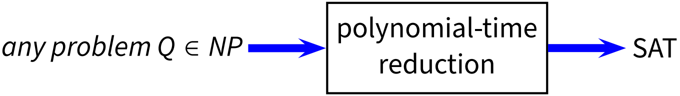

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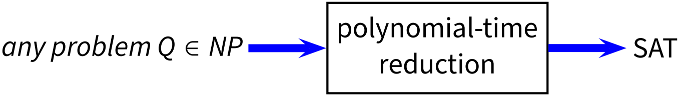

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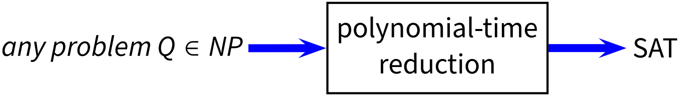

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- If a problem is NP-Hard (or NP-Complete) you should not feel so bad for not finding an efficient solution algorithm