Arithmetic Operations

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Outline

- Representing numbers
- Adding numbers
- Multiplying numbers

Representing Numbers

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  - ten symbols: 0, 1, 2, . . . , 9 (why ten?)

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$$n = d_6 b^6 + d_5 b^5 + \cdots + d_1 b^1 + d_0 b^0$$

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$$+5 \times 100 + 9 \times 10 + 2 \times 1 = 3141592$$

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- How do computers represent numbers?

Representing Numbers in a Computer

- Computers work well with the ***binary representation***
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$$n = 1 \times 64_{\text{ten}} + 1 \times 16_{\text{ten}} + 1 \times 8_{\text{ten}} + 1 \times 2_{\text{ten}} + 1 \times 1_{\text{ten}} = \ldots$$

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- The usual questions
  - Is this representation correct?
  - How much does it cost? (This time in terms of space)
  - Can we do better?

A *Ternary* Computer?

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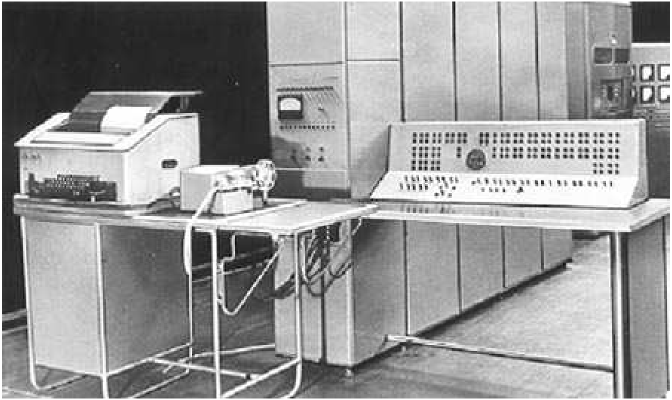
- A *ternary* computer was actually built!



The ***Setun*** was a *ternary* (or *trinary*) computer developed in 1958 and used at Moscow State University until about 1965

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- A representation should be unambiguous
  - i.e., there is *at most one* value for each representation
- We don’t care to say more about this

Space Complexity

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- Can we do better?
- No!
  - there are exactly b^ℓ combinations of ℓ symbols chosen from an alphabet Σ with $|\Sigma| = b$
  - i.e., you can not *express* more than b^ℓ values with ℓ symbols

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    3. $b \geq 3$, therefore $\log_b 3 \leq 1$, therefore $\ell \leq 2$

- We know that three base-b digits add up to a number of up to *two* base-b digits, so this is our building block

$$(x_i, y_i, z_i) \longrightarrow (c, s_i)$$

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- So, given x and y

$$\begin{array}{rcccc} x = & x_{\ell-1} & \ldots & x_1 & x_0 \\ y = & y_{\ell-1} & \ldots & y_1 & y_0 \\ x + y = & & & & \end{array}$$

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$R = 0$ // $[0, \ldots, 0]$ of size $\ell + 1$
$c = 0$
for $i = 1$ **to** ℓ
        $(c, R[i]) =$ **AddThreeDigits**$(A[i], B[i], c)$
$R[\ell + 1] = c$
return R

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- Is it correct? Yes

- How long does it take?

- Can we do better?

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- Can we do better? No!
  - we have to at least look at the ℓ symbols from the input values
  - we must assign at least $\ell + 1$ symbols for the result

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- Muliplying by b^i is equivalent to *shifting* our representation to the left by i positions
  - *left* means in the direction of the most significant bits

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- How long does it take?

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  which means $x = 2^{\ell/2}x_L + x_R$ and $y = 2^{\ell/2}y_L + y_R$, so…

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which, as we will see, leads to a much better complexity

$$T(\ell) = O(\ell^{\log_2 3}) = O(\ell^{1.59})$$