Greedy Algorithms

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Outline

- Greedy strategy
- Examples
- Activity selection
- Huffman coding

Recap on MST Algorithms

- Find the MST of $G = (V, E)$ with $w : E \longrightarrow \mathbb{R}$
  - find a $T \subseteq E$ that is a *minimum-weight spanning tree*

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```
GENERIC-MST(G, w)
A = ∅
while A is not a spanning tree
        find a safe edge e = (u, v) // the lightest that...
        A = A ∪ {e}
```

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   - i.e., that the greedy choice always leads to an optimal solution
     - not necessarily always the same one
3. Prove that the remaining subproblem is such that
   - combining the greedy choice with the optimal solution of the subproblem gives an optimal solution to the original problem

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  - not considering the results of the subproblems

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  optimal-substructure property: *an optimal solution to the problem contains within it optimal solutions to subproblems*

- It is natural to prove this by induction
  - if the solution to the subproblem is optimal, then combining the greedy choice with that solution yields an optimal solution

Example

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  - if $v(x_i) = \max_{x \in X} v(x)$ and A' is an optimal solution for $X' = X - \{x_i\}$, then $A' \subset A$

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- *Proving it optimal* may be difficult
  - requires deep understanding of the ***structure of the problem***

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  Optimal: $4 \times 1 + 2 \times 0.25 + 3 \times 0.1 = 4.8$ (9 coins/bills)

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  - v_i is the value of item i
  - w_i is the weight of item i
  - W is the maximum weight that the thief can carry

Problem: choose which items to take to maximize the total value of the robbery

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- **Exercise:**
  1. formulate a reasonable greedy choice
  2. prove that it doesn't work with a counter-example
  3. go back to (1) and repeat a couple of times

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- **Exercise:** prove that it is a greedy problem

Activity-Selection Problem

- A conference room is shared among different activities
  - $S = \{a_1, a_2, \ldots, a_n\}$ is the set of proposed activities
  - activity a_i has a *start time* s_i and a *finish time* f_i
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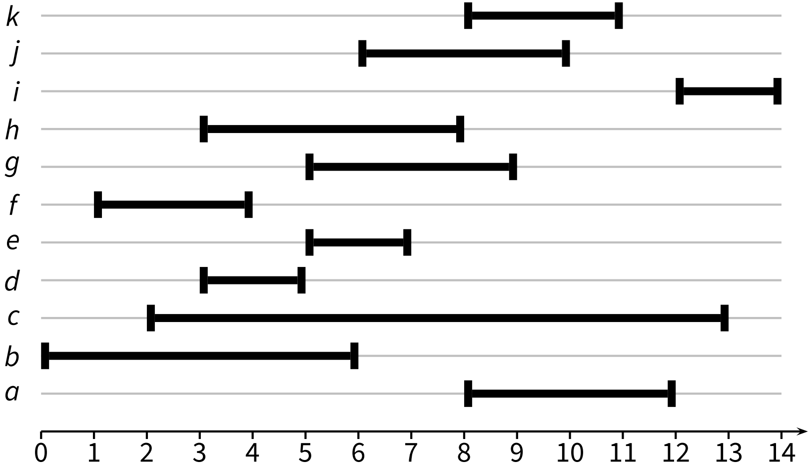
- Example

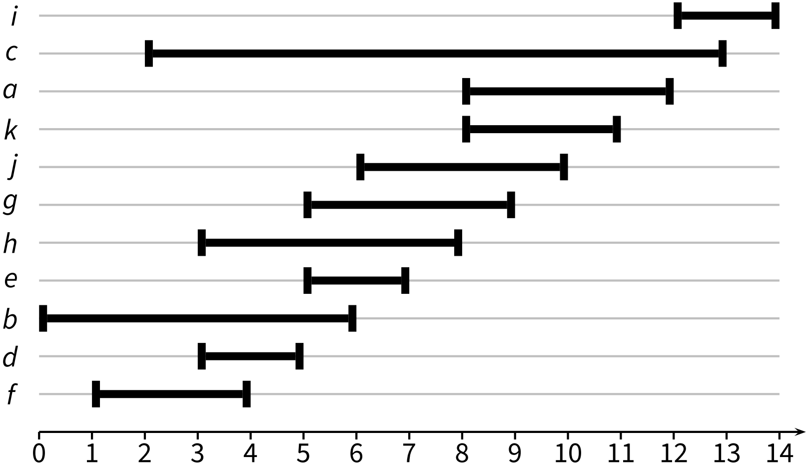
activity	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*
start	8	0	2	3	5	1	5	3	12	6	8
finish	12	6	13	5	7	4	9	8	14	10	11

- Is there a greedy solution for this problem?

Activity-Selection Problem (2)



Activity-Selection Problem (3)



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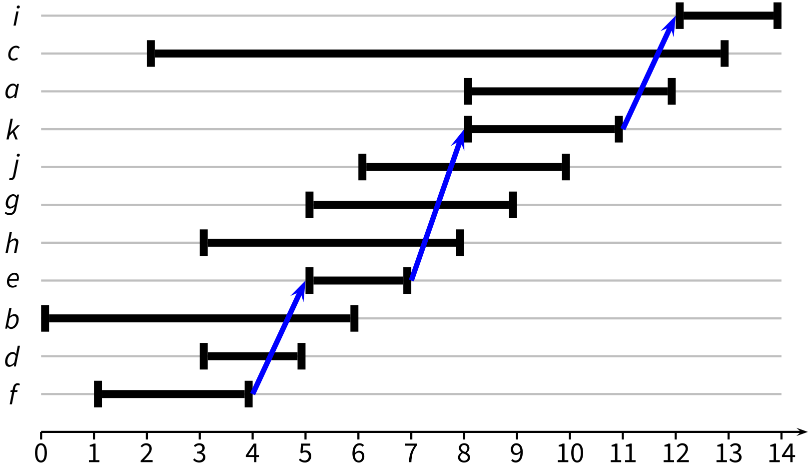

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  - thus OPT^* is an *optimal* solution, because $|OPT^*| = |OPT|$

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  - by construction, $\overline{S} \subseteq S'$, so $OPT \setminus \{a_m\}$ is a solution also for S'
  - which means that there is a solution S' of size $|OPT| - 1$, which contradicts the main assumption that $|OPT'| < |OPT| - 1$

- Suppose you have a large sequence S of the six characters: ‘a’, ‘b’, ‘c’, ‘d’, ‘e’, and ‘f’
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- Can we do better?

Huffman Coding (2)

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symbol	*code*
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b	001
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- *Observation:* the encoding of ‘e’ and ‘f’ is a bit redundant
  - the second bit does not help us in distinguishing ‘e’ from ‘f’
  - in other words, if the first (most significant) bit is 1, then the second bit gives us no information, so it can be removed

Idea

- Variable-length code

symbol	*code*
a	000
b	001
c	010
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- Encoding and decoding are well-defined and unambiguous

- Variable-length code

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Idea

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- Given the frequencies $f_a, f_b, f_c, \ldots$ of all the symbols in S

$$M = 3n(f_a + f_b + f_c + f_d) + 2n(f_e + f_f)$$

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- The average codeword size

$$B(S) = \sum_{c \in C} f(c)|E(c)|$$

is minimal

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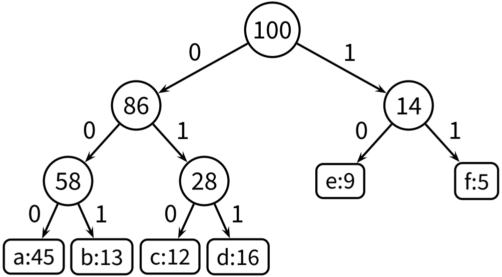

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  - the weight $f(v)$ of a node v is the frequency of its code/prefix

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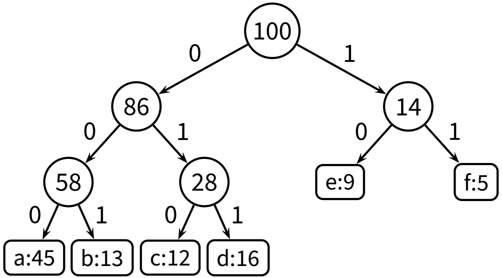

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$$B(S) = n \sum_{c \in leaves(T)} f(c) depth(c) = n \sum_{v \in T} f(v)$$

Huffman Algorithm

```
HUFFMAN(C)
n = |C|
Q = C
for i = 1 to n − 1
        create a new node z
        z.left = EXTRACT-MIN(Q)
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        f(z) = f(z.left) + f(z.right)
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- We build the code bottom-up
- Each time we make the “greedy” choice of merging the two least frequent nodes (symbols or prefixes)

Example

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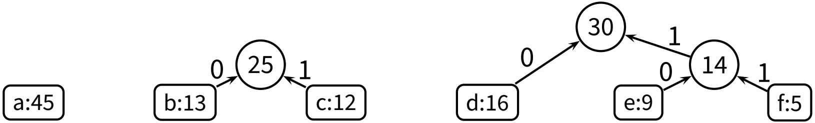

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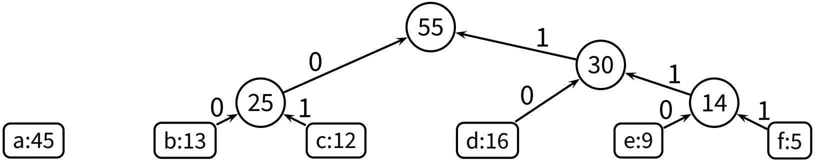

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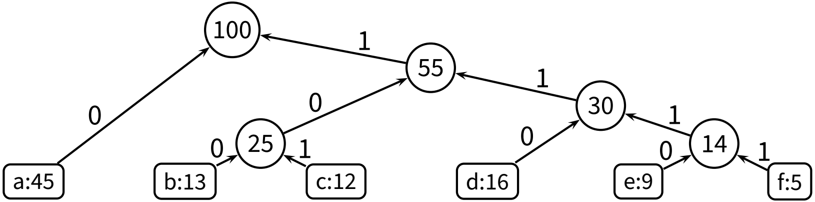

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```

sym.	*freq.*	*code*
a	45%	0
b	13%	100
c	12%	101
d	16%	110
e	9%	1110
f	5%	1111