Elementary Data Structures and Hash Tables

Antonio Carzaniga

Faculty of Informatics
Università della Svizzera italiana

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Outline

- Common concepts and notation
- Stacks
- Queues
- Linked lists
- Trees
- Direct-access tables
- Hash tables

Concepts

- A ***data structure*** is a way to organize and store information
  - to facilitate access, or for other purposes

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- A data structure stores ***data*** and possibly ***meta-data***
  - e.g., a *heap* needs an array A to store the keys, plus a variable $A.\mathit{heap\text{-}size}$ to remember how many elements are in the heap

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  - **STACK-EMPTY**(S) returns TRUE if and only if S is empty
  - **PUSH**(S, x) pushes the value x onto the stack S
  - **POP**(S) extracts and returns the value on the top of the stack S

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- *Implementation*
  - using an array
  - using a linked list
  - $\ldots$

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if S.top == 0
    return TRUE
else return FALSE
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STACK-EMPTY(S)

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if S.top == 0
        return TRUE
else return FALSE
```

PUSH(S,x)

```
S.top = S.top + 1
S[S.top] = x
```

POP(S)

```
if STACK-EMPTY(S)
        error "underflow"
else S.top = S.top − 1
        return S[S.top + 1]
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Queue

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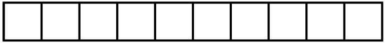
- *Implementation*
  - Q is an array of fixed length $Q.length$
    - i.e., Q holds at most $Q.length$ elements
    - enqueueing more than Q elements causes an “overflow” error
  - $Q.head$ is the position of the “head” of the queue
  - $Q.tail$ is the first empty position at the tail of the queue

Enqueue

```
ENQUEUE(Q,x)
if Q.queue-full
        error "overflow"
else Q[Q.tail] = x
        if Q.tail < Q.length
                Q.tail = Q.tail + 1
        else Q.tail = 1
        if Q.tail == Q.head
                Q.queue-full = TRUE
        Q.queue-empty = FALSE
```



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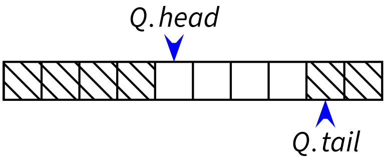

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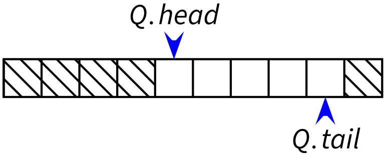

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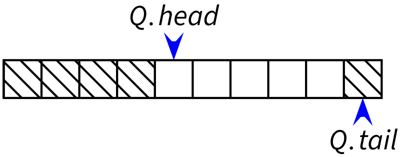

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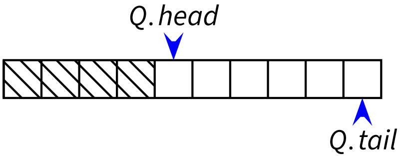

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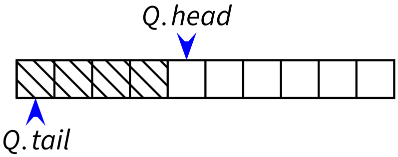

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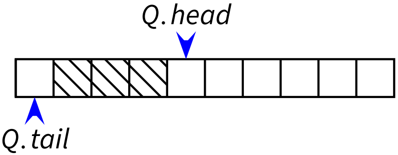

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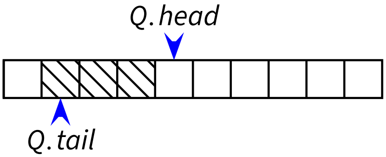

Dequeue

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DEQUEUE(Q)
if Q.queue-empty
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else x = Q[Q.head]
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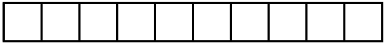

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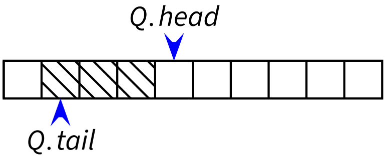

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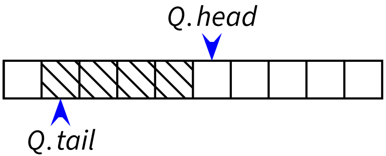

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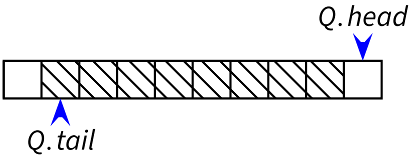

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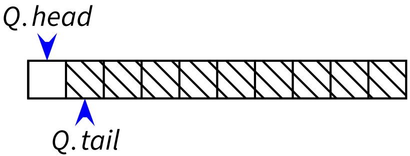

- *Interface*
  - **List-Insert**(L, x) adds element x at beginning of a list L
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  - **List-Search**(L, k) finds an element whose key is k in a list L

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- *Implementation*
  - a *doubly-linked* list
  - each element x has two "links" $x.prev$ and $x.next$ to the previous and next elements, respectively
  - each element x holds a key $x.key$
  - it is convenient to have a dummy "sentinel" element $L.nil$

Linked List With a “Sentinel”

```
LIST-INIT(L)
L.nil.prev = L.nil
L.nil.next = L.nil
```

```
LIST-INSERT(L, x)
x.next = L.nil.next
L.nil.next.prev = x
L.nil.next = x
x.prev = L.nil
```

```
LIST-SEARCH(L, k)
x = L.nil.next
while x ≠ L.nil ∧ x.key ≠ k
        x = x.next
return x
```

Algorithm	*Complexity*

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Stack-Empty	

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POP	$O(1)$
ENQUEUE	$O(1)$
DEQUEUE	$O(1)$
LIST-INSERT	$O(1)$
LIST-DELETE	$O(1)$
LIST-SEARCH	$\Theta(n)$

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  - ***hash tables***

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```
DIRECT-ADDRESS-INSERT(T, k)
1  T[k] = TRUE
```

```
DIRECT-ADDRESS-DELETE(T, k)
1  T[k] = FALSE
```

```
DIRECT-ADDRESS-SEARCH(T, k)
1  return T[k]
```

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- The ***space complexity*** is $\Theta(|U|)$
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Direct-Address Table (2)

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- *Can we have the benefits of a direct-address table but with a table of reasonable size?*

- *Idea*
  - use a table T with $|T| \ll |U|$
  - map each key $k \in U$ to a position in T, using a ***hash function***

$$h : U \longrightarrow \{1, \ldots, |T|\}$$

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HASH-INSERT(T, k)
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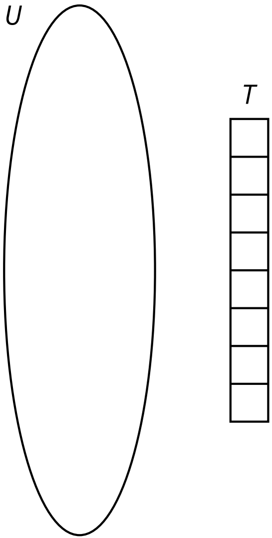
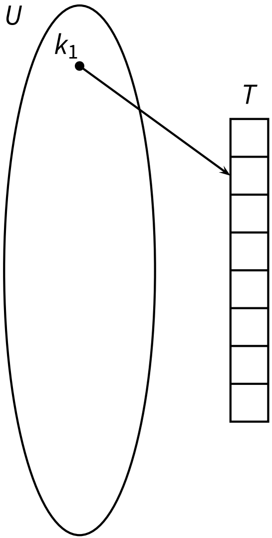
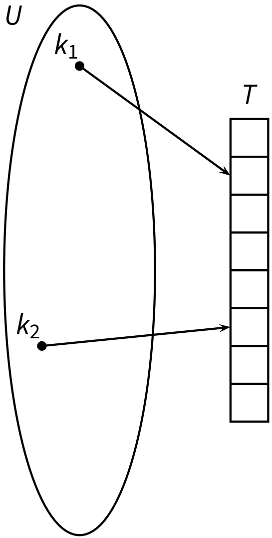
Are these algorithms correct? No!

What if two distinct keys $k_1 \neq k_2$ *collide?* (I.e., $h(k_1) = h(k_2)$)

Hash Table

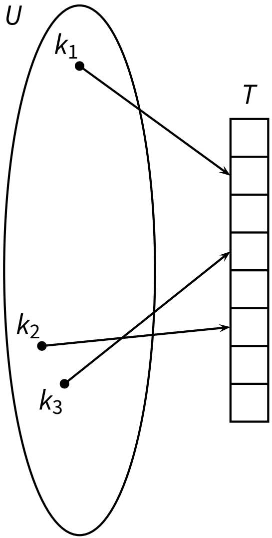


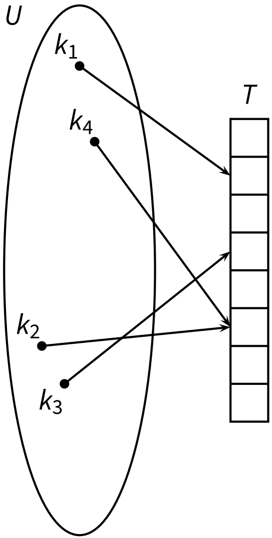

U
k_1
T

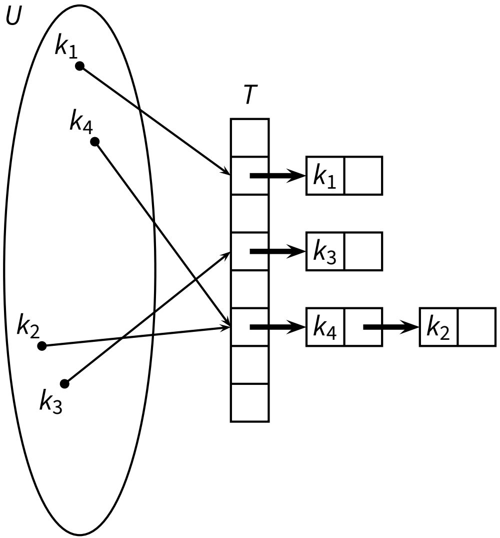


U
k_1
T
k_2

Hash Table




U
k_1
k_4
T
k_2
k_3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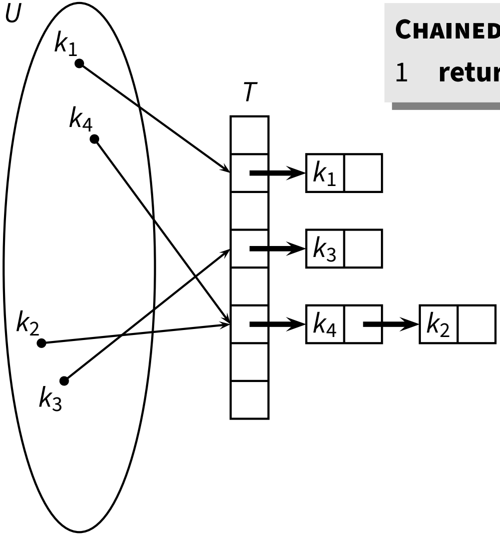
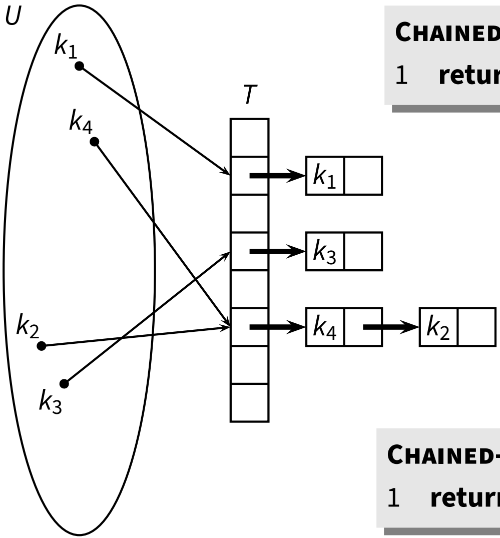


CHAINED-HASH-INSERT(T, k)

1 **return LIST-INSERT**$(T[h(k)], k)$



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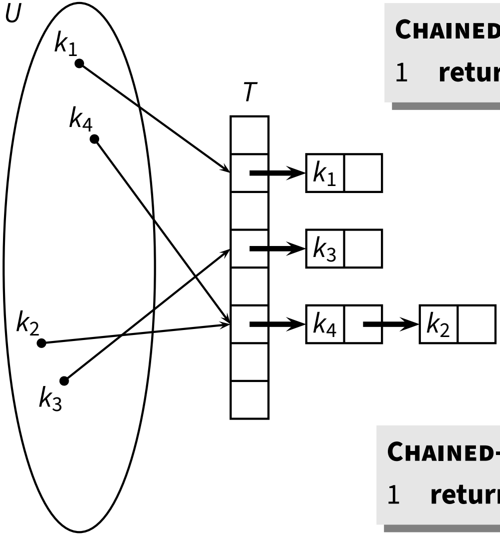
Hash Table



CHAINED-HASH-INSERT(T, k)

```
return LIST-INSERT(T[h(k)], k)
```

load factor

$$\alpha = \frac{n}{|T|}$$

CHAINED-HASH-SEARCH(T, k)

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(The formalism is actually a bit more complicated.)

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$$E[n_i] = \frac{n}{|T|} = \alpha$$

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- We further assume that $h(k)$ can be computed in $O(1)$ time

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- So, given n distinct keys, the expected length n_i of the linked list at position i is

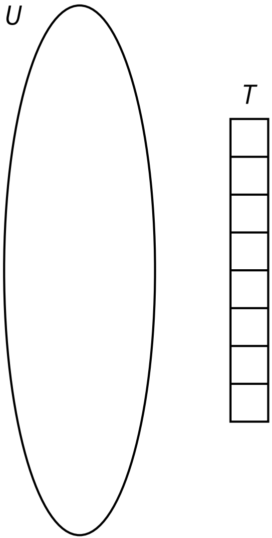
$$E[n_i] = \frac{n}{|T|} = \alpha$$

- We further assume that $h(k)$ can be computed in $O(1)$ time

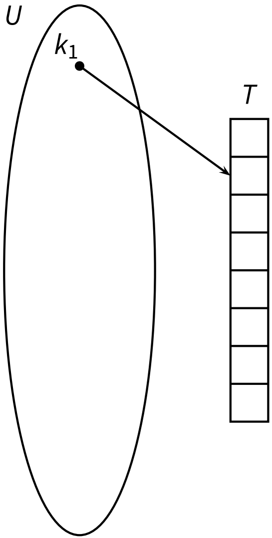
- Therefore, the complexity of **CHAINED-HASH-SEARCH** is

$$\boxed{\Theta(1 + \alpha)}$$

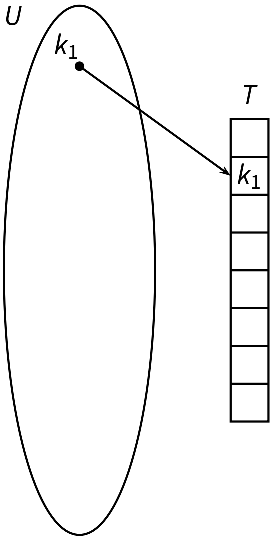
Open-Address Hash Table



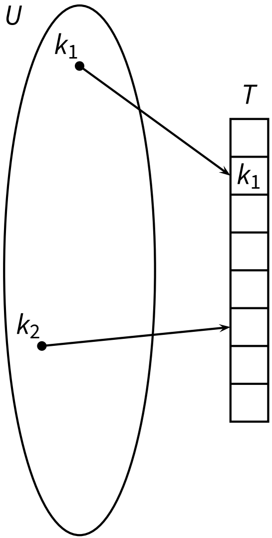
Open-Address Hash Table



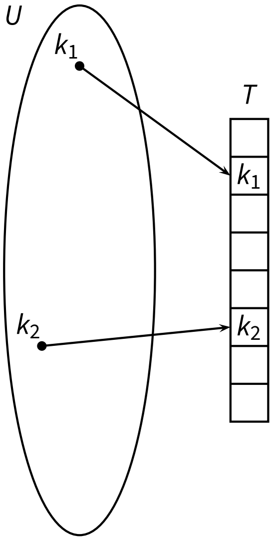
Open-Address Hash Table



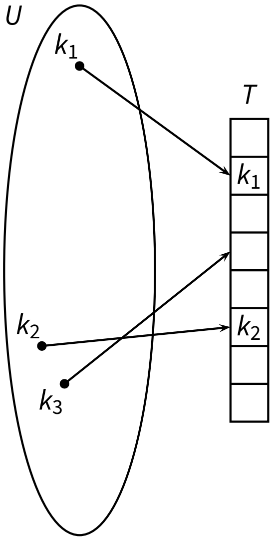
Open-Address Hash Table



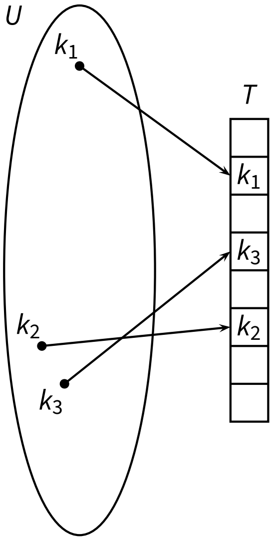
Open-Address Hash Table



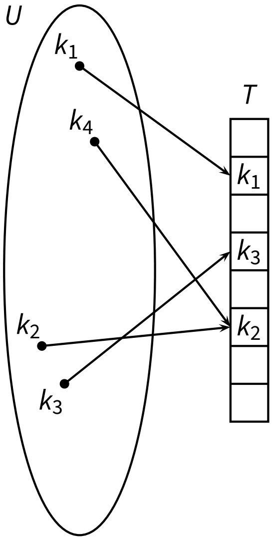
Open-Address Hash Table



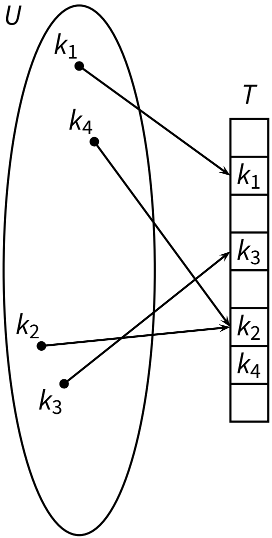
Open-Address Hash Table



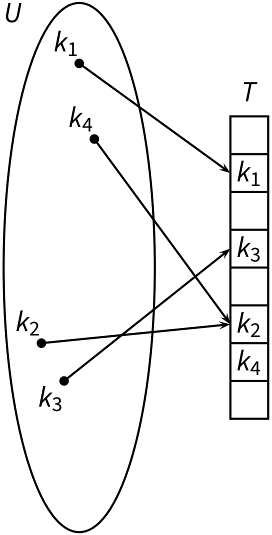
Open-Address Hash Table



Open-Address Hash Table



Open-Address Hash Table




```
HASH-INSERT(T, k)
j = h(k)
for i = 1 to T.length
        if T[j] == NIL
                T[j] = k
                return j
        elseif j < T.length
                j = j + 1
        else j = 1
error "overflow"
```

Open-Addressing (2)

- *Idea:* instead of using linked lists, we can store all the elements in the table
  - this implies $\alpha \leq 1$

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  - this implies $\alpha \leq 1$
- When a collision occurs, we simply find another free cell in T
- A sequential “probe” may not be optimal
  - can you figure out why?

Open-Addressing (3)

```
HASH-INSERT(T, k)
for i = 1 to T.length
j = h(k, i)
        if T[j] == NIL
                T[j] = k
                return j
error "overflow"
```

```
HASH-INSERT(T, k)
for i = 1 to T.length
    j = h(k, i)
    if T[j] == NIL
        T[j] = k
        return j
error "overflow"
```

- Notice that $h(k, \cdot)$ must be a ***permutation***
  - i.e., $h(k, 1), h(k, 2), \ldots, h(k, |T|)$ must cover the entire table T