Basic Elements of Complexity Theory

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Outline

- Basic complexity classes
- Polynomial reductions
- NP-completeness

Polynomial Time

- A ***polynomial-time algorithm*** is one whose worst-case running time $T(n)$, on an input of size n bits, is $O(n^k)$ for some *constant* k

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  all pairs

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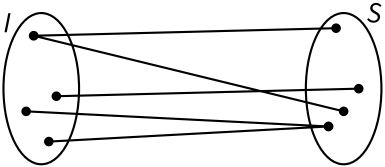
polynomial $\equiv$ *good*

super-polynomial $\equiv$ *bad*

Problems

- A ***problem*** Q is a binary relation between a set I of ***instances*** and a set S of ***solutions***



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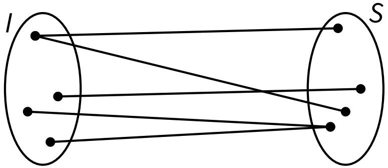

- A ***concrete problem*** Q is one where I and S are the set of binary strings $\{0, 1\}^*$
  - for all practical purposes, instances and solutions can be ***encoded*** as binary strings (i.e., mapped into $\{0, 1\}^*$)
  - we consider only sensible encodings…

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Primality Testing

Decision vs. Optimization: Example

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- Shortest path in a graph

$$G = (V = \{a, b, c, \ldots\}, E = \{(a, c), \ldots\}), a, z \longrightarrow a, c, \ldots, z$$

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- *Shortest path as a* ***decision problem***

$$G = (V = \{a, b, c, \ldots\}, E = \{(a, c), \ldots\}), a, z, 10 \longrightarrow 1$$

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  - *input:* a graph G, a start vertex (a), an end vertex (z), and a path length (10)
  - *output:* 1 if there is a path of (at most) the given length

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- An optimization problem is ***at least as hard*** as its corresponding decision problem
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- An optimization problem is ***not much harder*** than the corresponding decision problem
  - having a solution to the decision problem does not give an immediate solution to the optimization problem
  - but we can typically use the decision problem as a subroutine in some kind of (binary) search to solve the corresponding optimization problem

The Complexity Class P

- A concrete decision problem *Q* is ***polynomial-time solvable*** if there is a polynomial-time algorithm *A* that solves it

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  - primality

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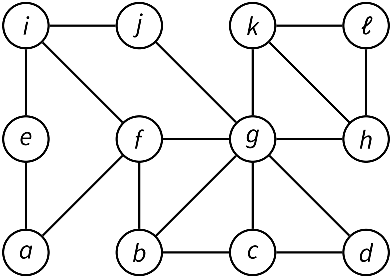

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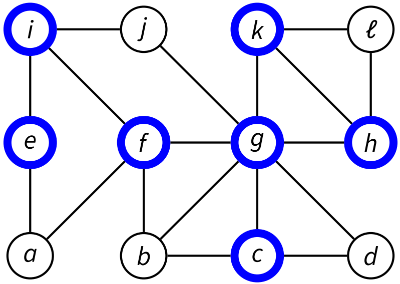

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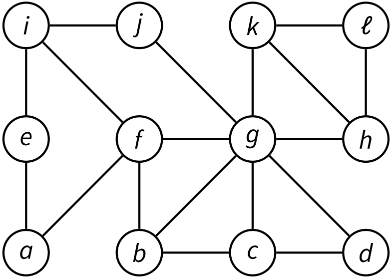

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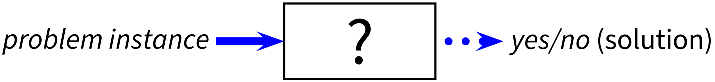

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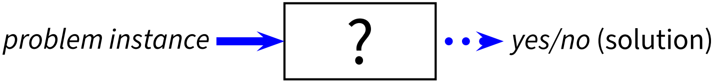

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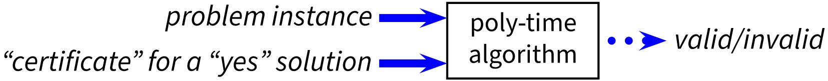

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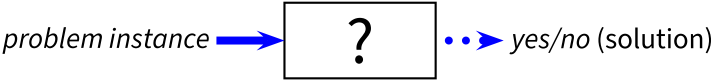

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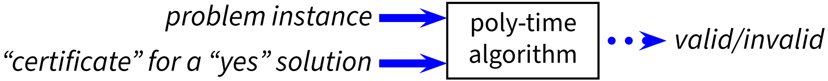

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$$P \subseteq NP$$

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Finding a solution to a problem is believed to be inherently more difficult than verifying a given solution (or a proof of a solution)

. . . but nobody has been able to prove that this is the case!

Example: SAT

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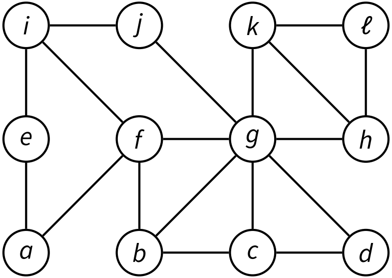

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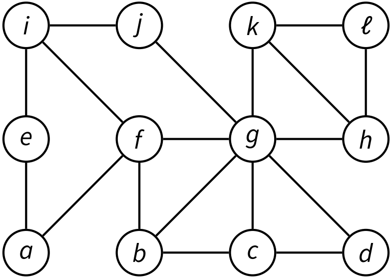

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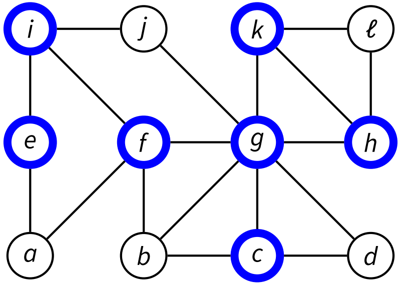

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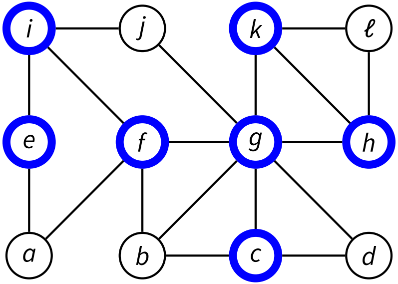

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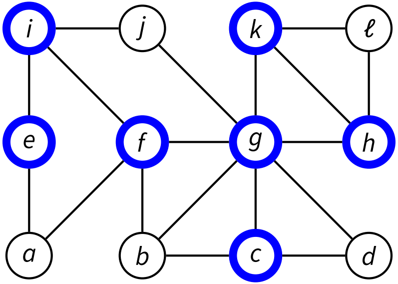

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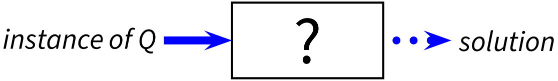

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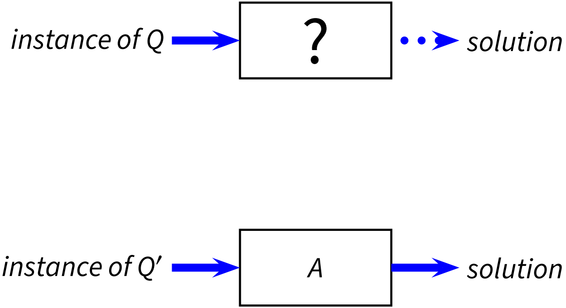

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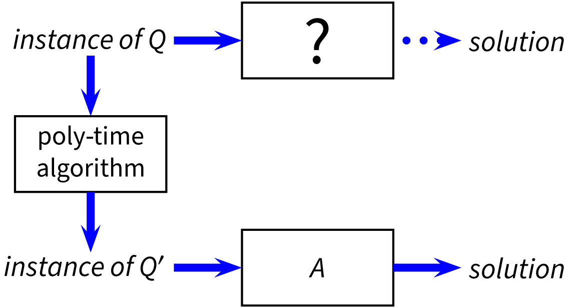

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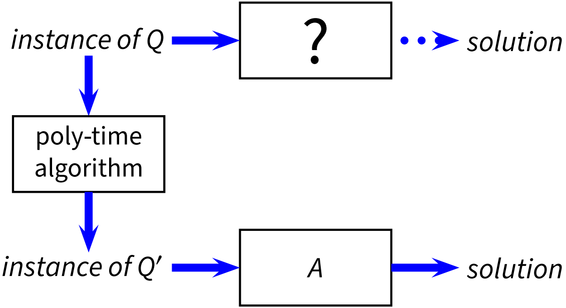

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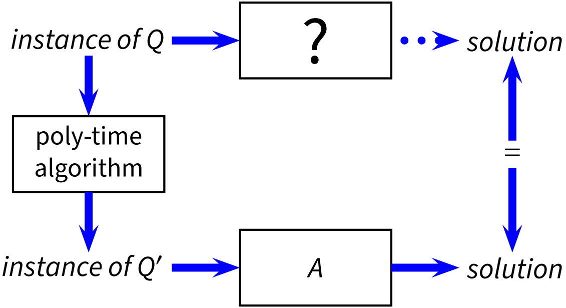

  - an instance q of Q is transformed into an instance q' of Q' through a polynomial-time algorithm
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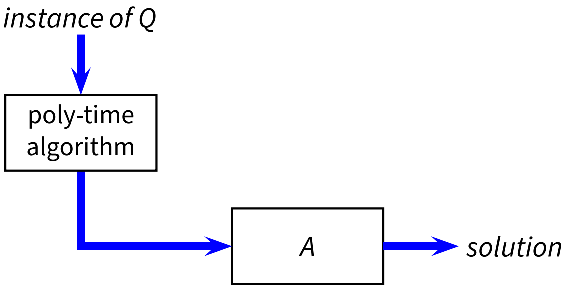
- Solution by polynomial-time reductions to a solvable problem



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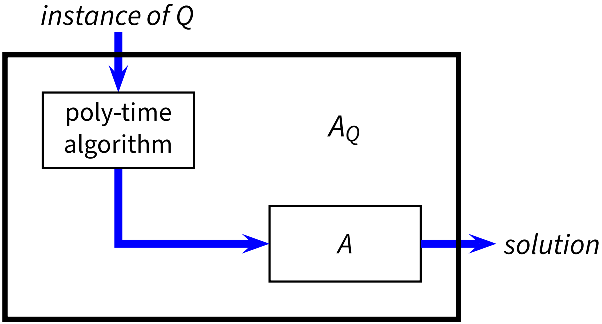

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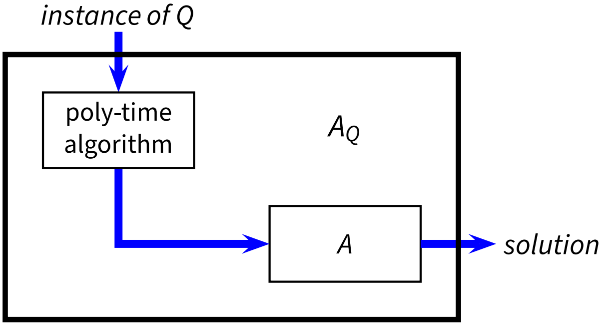

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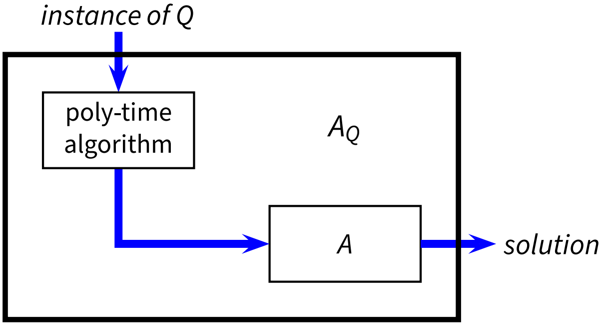

- if A is polynomial-time, then of A_Q is also polynomial time
- therefore if $Q' \in P$, then $Q \in P$

Example: 2-CNF-SAT

2-CNF-SAT problem

Input:

- f is a Boolean formula of n (Boolean) variables $x_1, x_2, \ldots, x_n$
- f is in *conjunctive normal form (CNF)*, so $f = C_1 \wedge C_2 \wedge \cdots \wedge C_k$
- every *clause* C_i of f contains exactly *two* literals (a variable or its negation)

Output: 1 iff f is satisfiable

- there is an assignment of variables that satisfies f

Example:

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2)$$

2-CNF-SAT to Implicative Form

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$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3)$$

is equivalent to

$$(\neg x_1 \Rightarrow \neg x_3) \wedge (x_3 \Rightarrow x_1) \wedge (x_2 \Rightarrow x_3) \wedge (\neg x_3 \Rightarrow \neg x_2)$$

2-CNF-SAT to Graph Reachability

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2-CNF-SAT to Graph Reachability

$$(x_1 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_3) \land (x_1 \lor x_2)$$

$$\Downarrow\Uparrow$$

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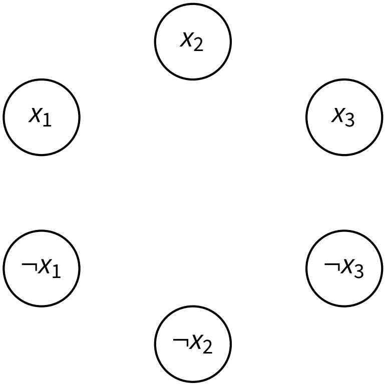

2-CNF-SAT to Graph Reachability

$$(x_1 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_3) \land (x_1 \lor x_2)$$

$$\Downarrow\Uparrow$$

$$(\neg x_1 \Rightarrow \neg x_3) \land (x_3 \Rightarrow x_1) \land (x_2 \Rightarrow x_3) \land (\neg x_3 \Rightarrow \neg x_2) \land$$
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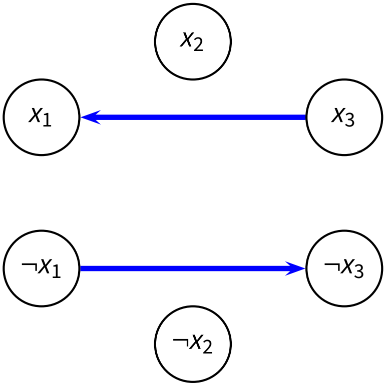

2-CNF-SAT to Graph Reachability

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$$(\neg x_1 \Rightarrow \neg x_3) \land (x_3 \Rightarrow x_1) \land (x_2 \Rightarrow x_3) \land (\neg x_3 \Rightarrow \neg x_2) \land$$
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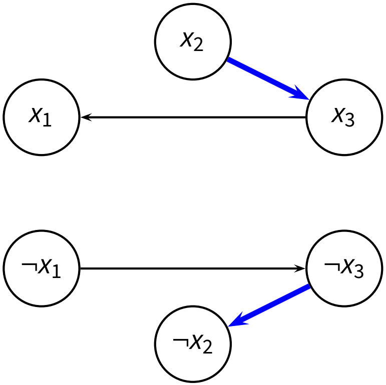

2-CNF-SAT to Graph Reachability

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2)$$

$$\Downarrow\Uparrow$$

$$(\neg x_1 \Rightarrow \neg x_3) \wedge (x_3 \Rightarrow x_1) \wedge (x_2 \Rightarrow x_3) \wedge (\neg x_3 \Rightarrow \neg x_2) \wedge$$
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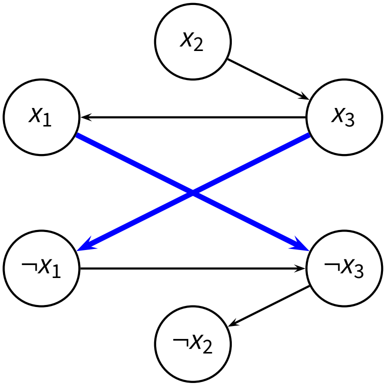

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge \boxed{(x_1 \vee x_2)}$$

$$\Downarrow\Uparrow$$

$$(\neg x_1 \Rightarrow \neg x_3) \wedge (x_3 \Rightarrow x_1) \wedge (x_2 \Rightarrow x_3) \wedge (\neg x_3 \Rightarrow \neg x_2) \wedge$$
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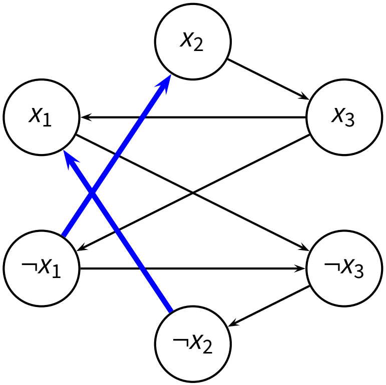

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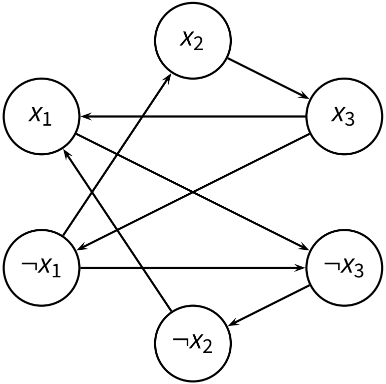

not satisfiable
if and only if
$x_i \rightsquigarrow \neg x_i \rightsquigarrow x_i$
for some i

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$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2)$$

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$$(\neg x_1 \Rightarrow \neg x_3) \wedge (x_3 \Rightarrow x_1) \wedge (x_2 \Rightarrow x_3) \wedge (\neg x_3 \Rightarrow \neg x_2) \wedge$$
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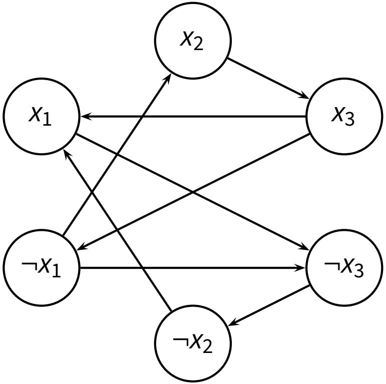

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↓

depth-first search

Reduction of 2-CNF-SAT

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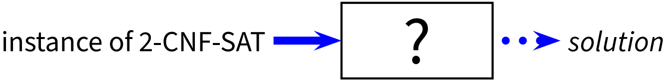
- 2-CNF-SAT $\in P$



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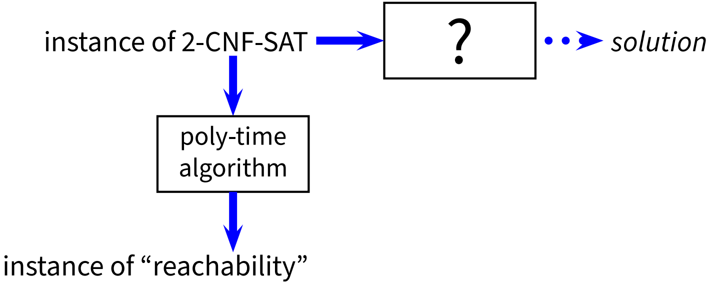
instance of 2-CNF-SAT
?
solution
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instance of "reachability"

Reduction of 2-CNF-SAT

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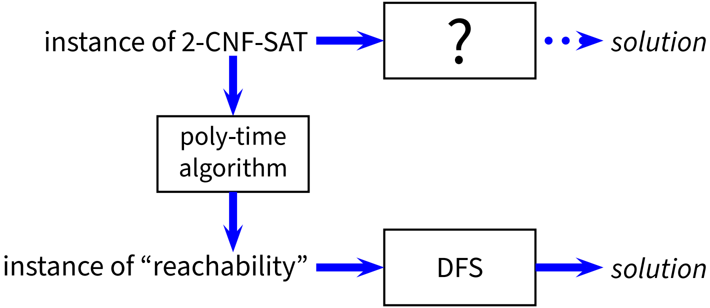

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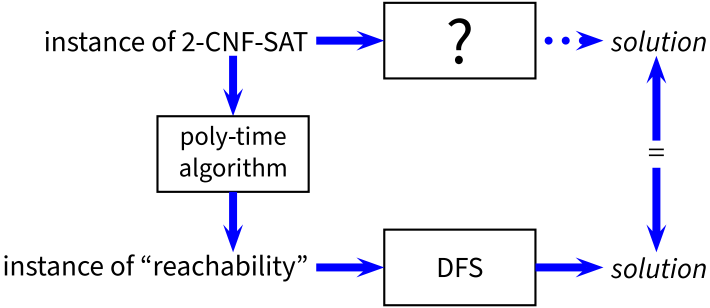
instance of 2-CNF-SAT
?
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=
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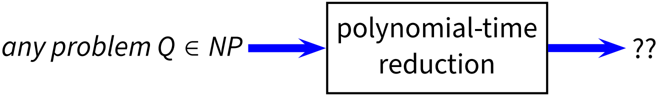
- If Q' is NP-hard and *polynomial-time solvable*, then P $=$ NP
  - most researchers believe that there is no such Q'

The First NP-Complete Problem

- Is there any NP-complete problem?



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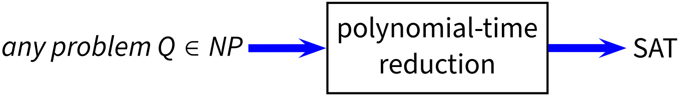

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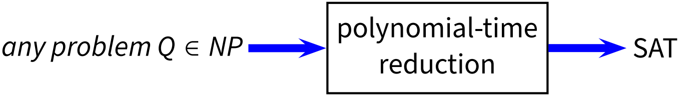

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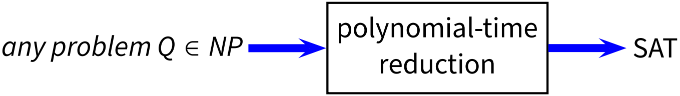

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- If a problem is NP-Hard (or NP-Complete) you should not feel so bad for not finding an efficient solution algorithm