

Dynamic Programming

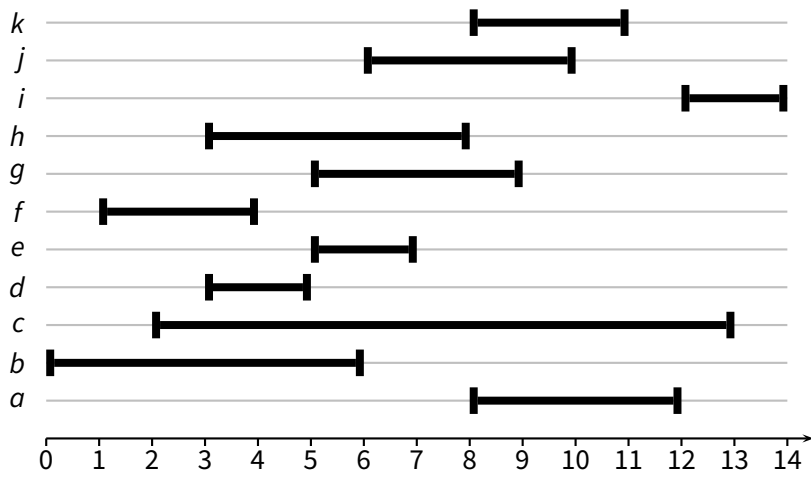
Antonio Carzaniga

Faculty of Informatics
Università della Svizzera italiana

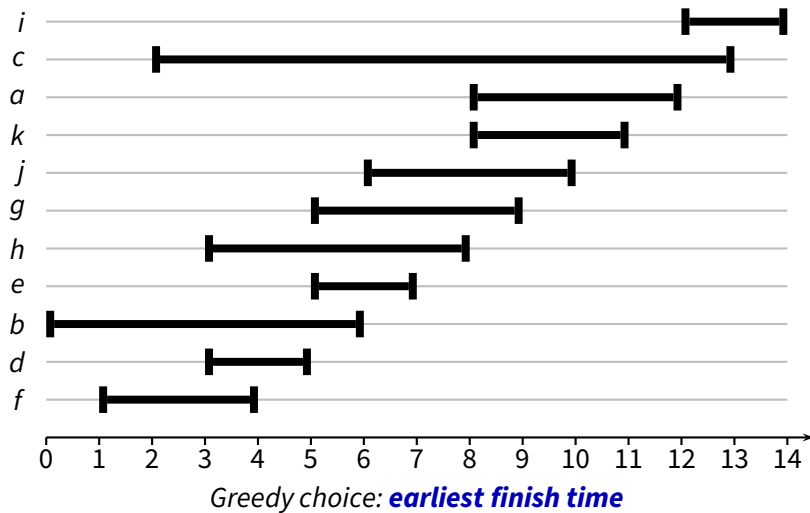
May 24, 2022

- Examples
- Dynamic programming strategy
- More examples

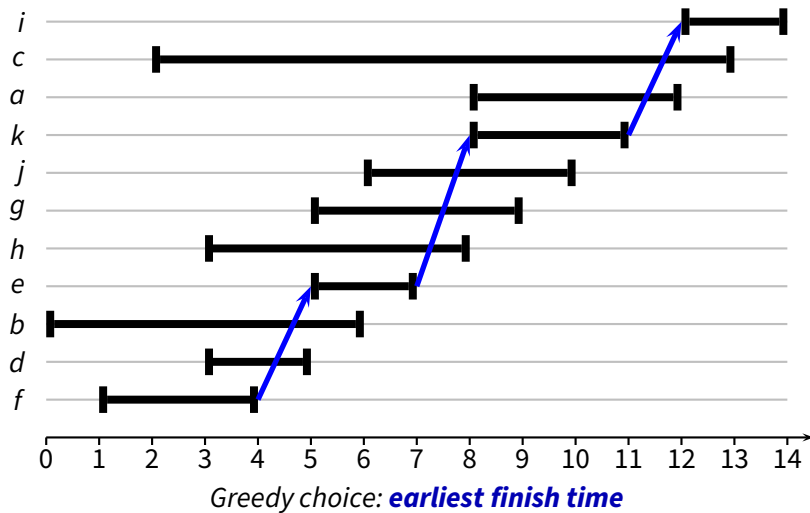
Activity-Selection Problem



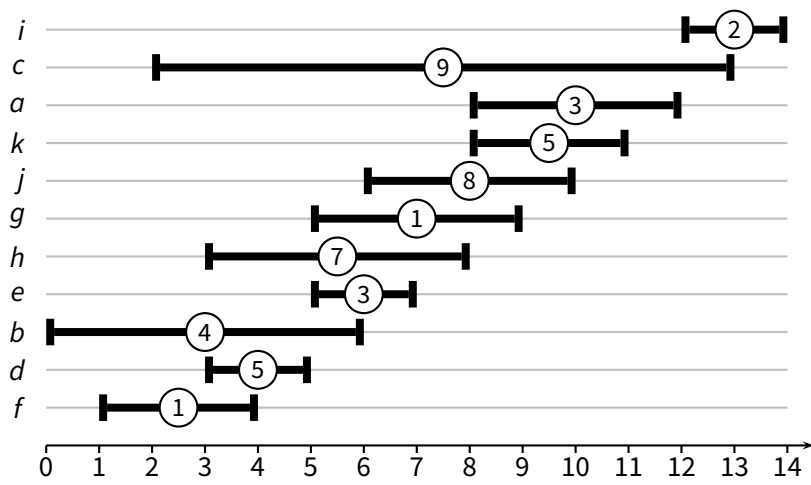
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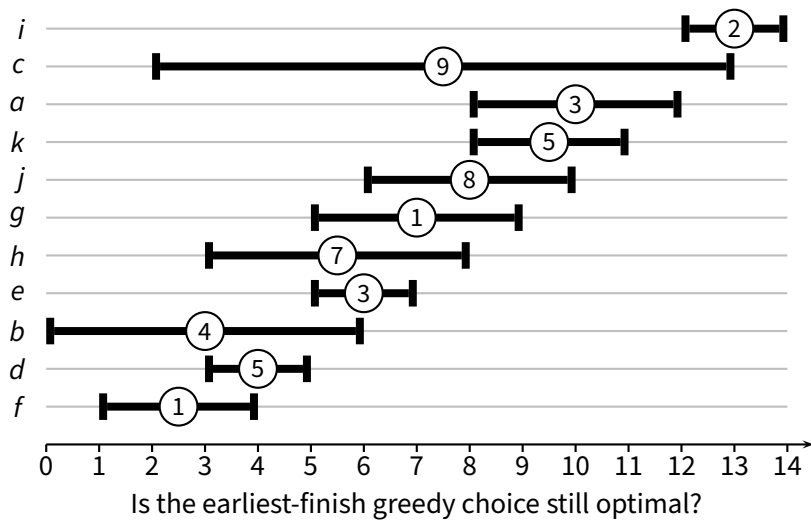
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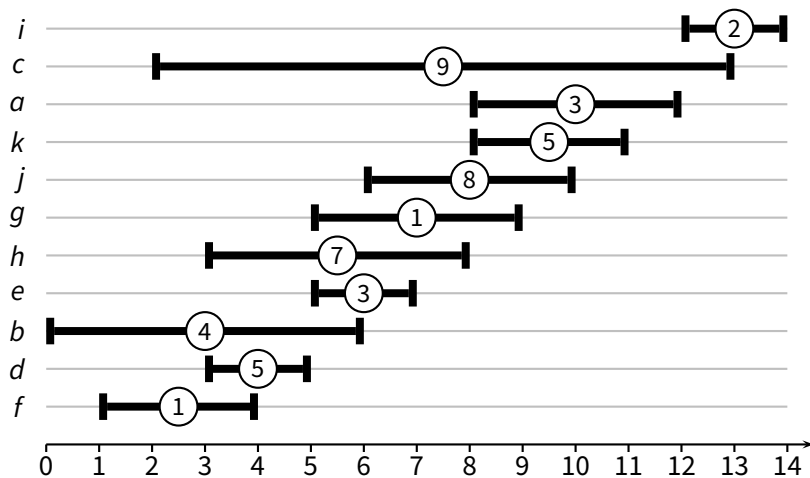
Weighted Activity-Selection Problem



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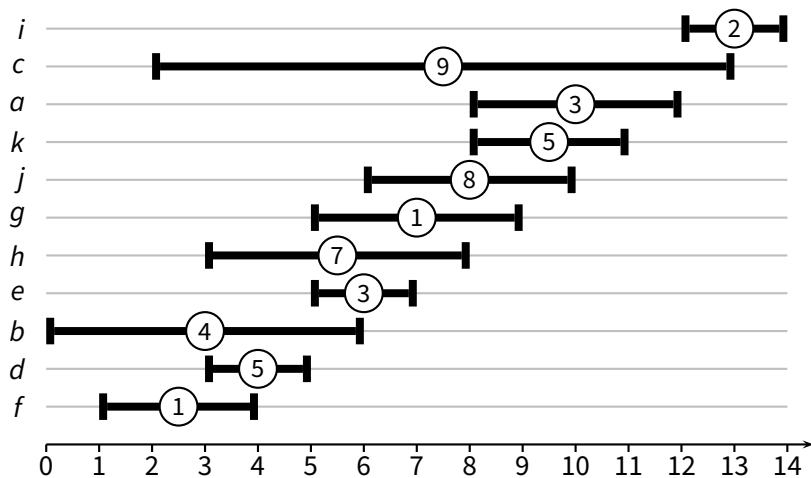


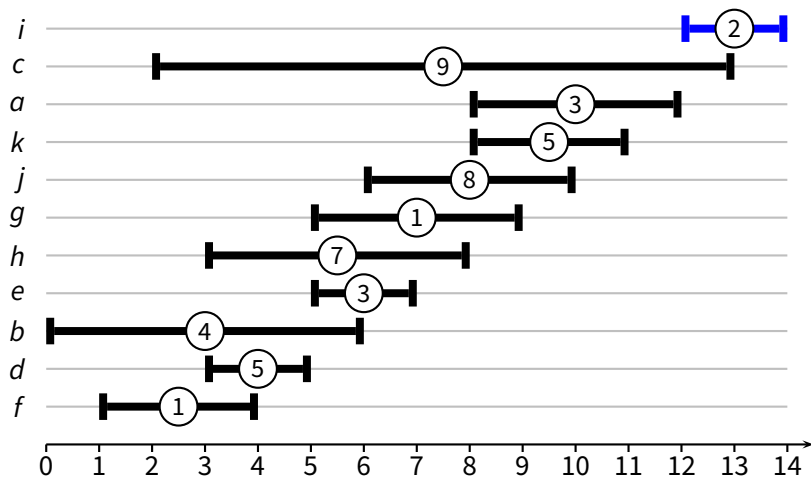
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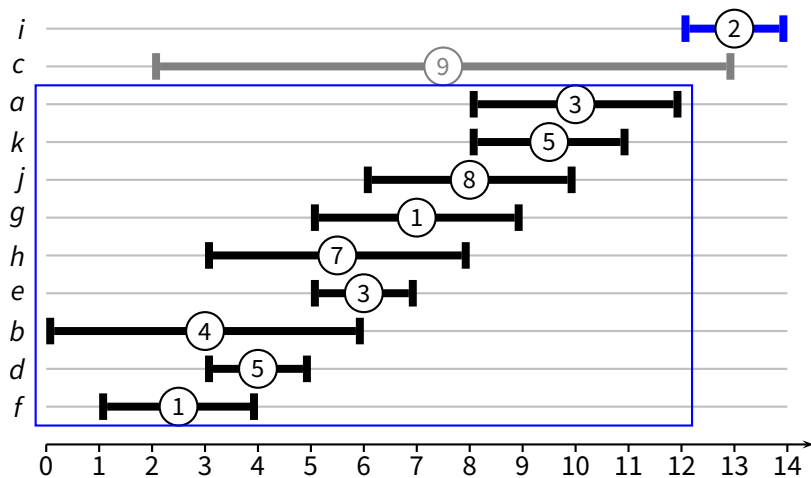
Is the earliest-finish greedy choice still optimal?

Is *any* greedy choice optimal?

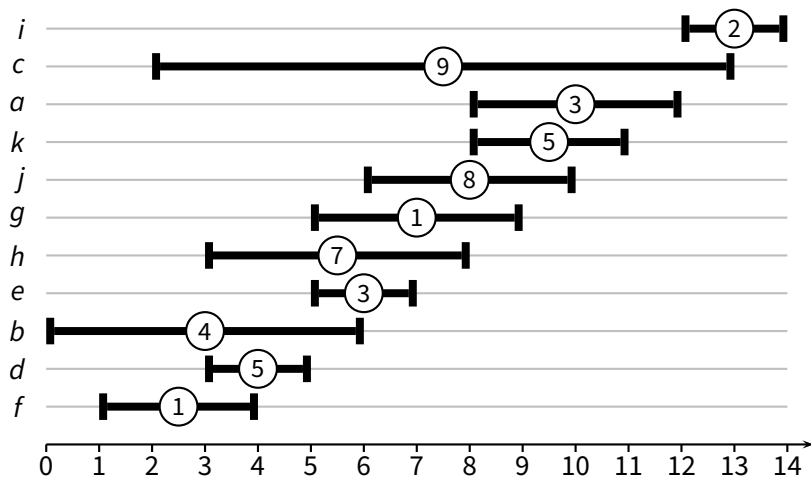


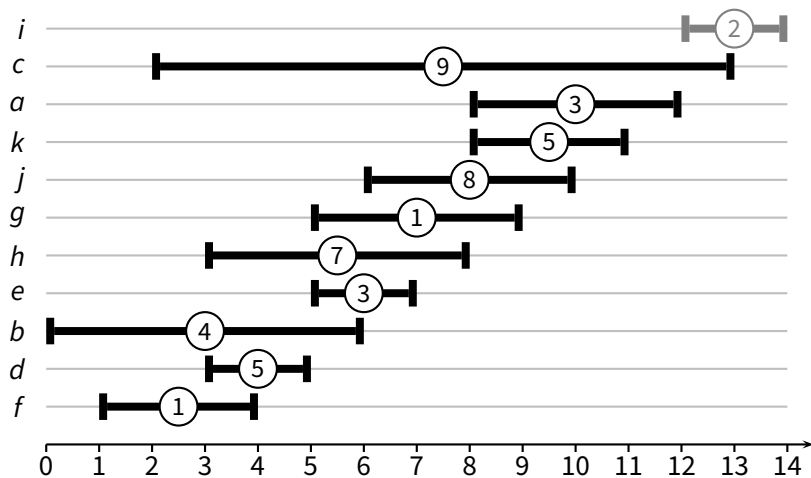


Case 1: activity *i* is in the optimal schedule

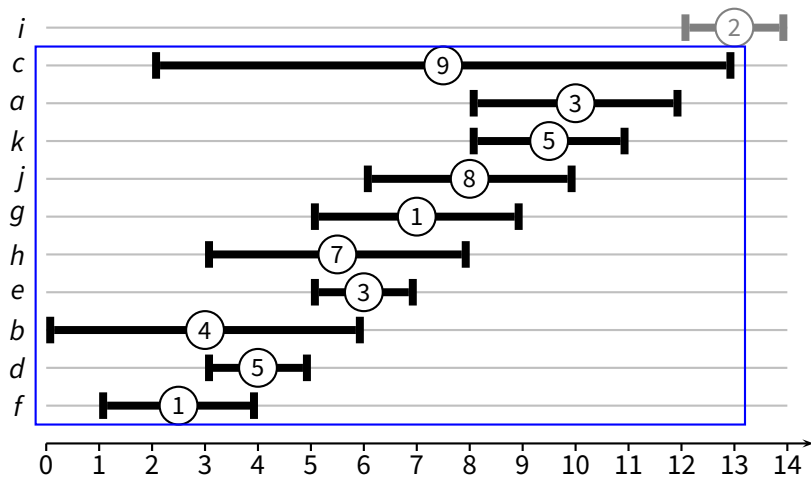


Case 1: activity i is in the optimal schedule





Case 2: activity *i* is not in the optimal schedule



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Bellman-Ford Algorithm

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- Given a graph $G = (V, E)$ and a weight function w , we compute the shortest distance $D_u(v)$, from $u \in V$ to $v \in V$, using the *Bellman-Ford equation*

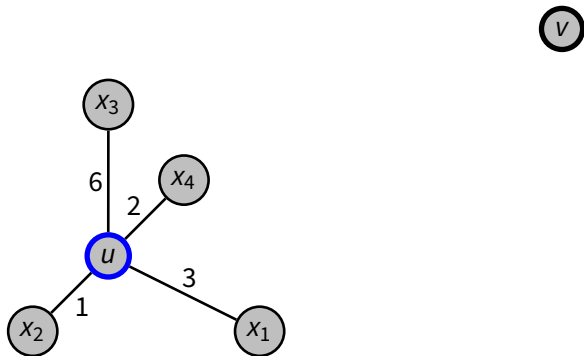
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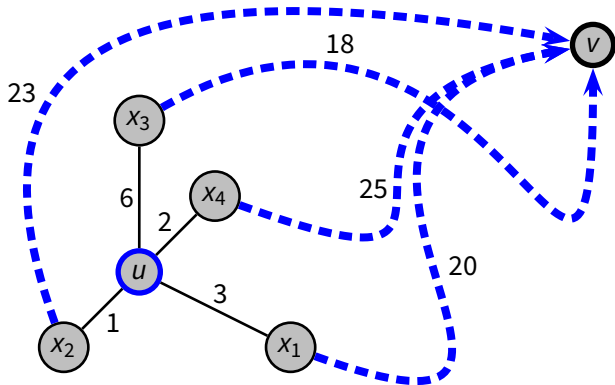
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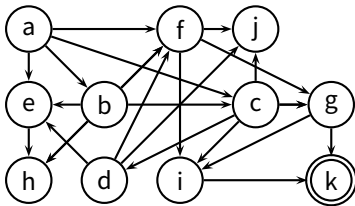
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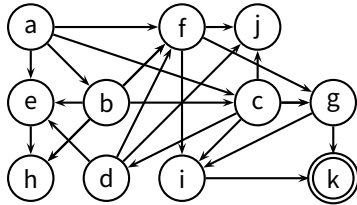
Shortest Paths on DAGs

- Given a *directed acyclic graph* $G = (V, E)$, this one with unit weights, find the shortest distances to a given node

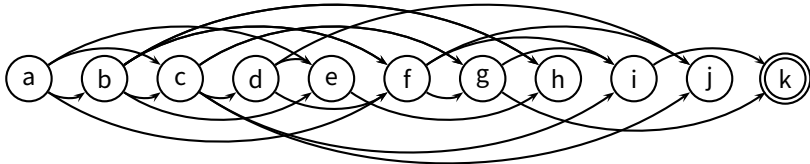


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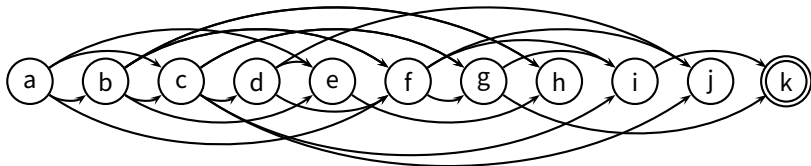
- Considering V in **topological order**...



Shortest Paths on DAGs (2)

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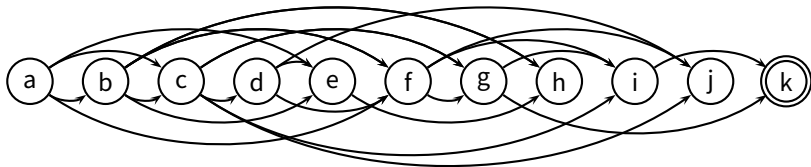
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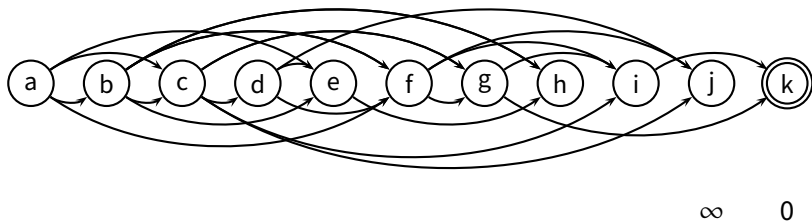


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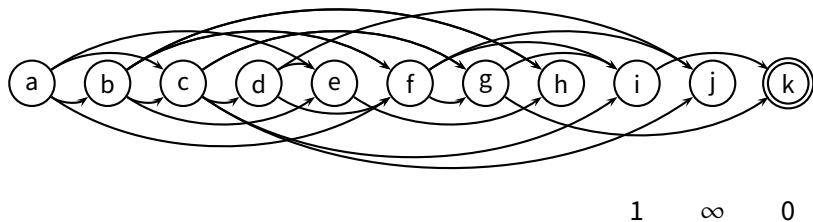
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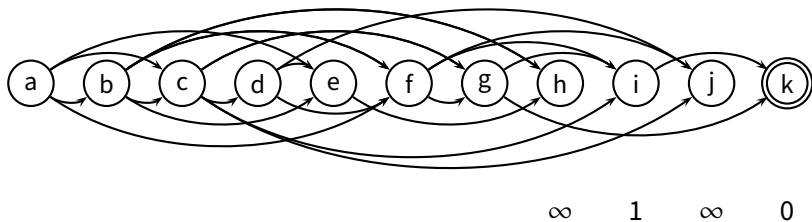
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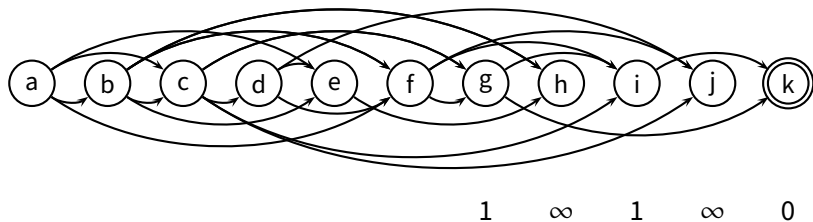
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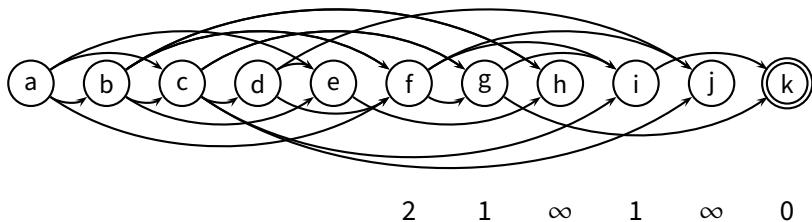
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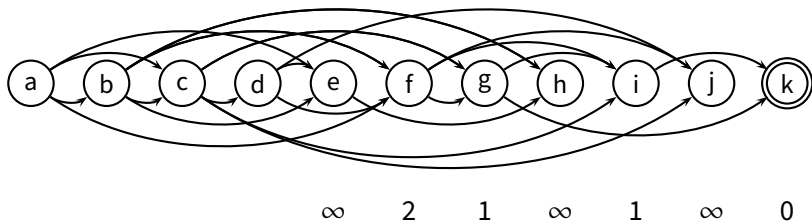
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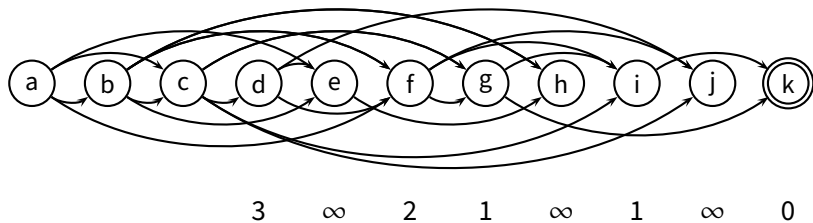
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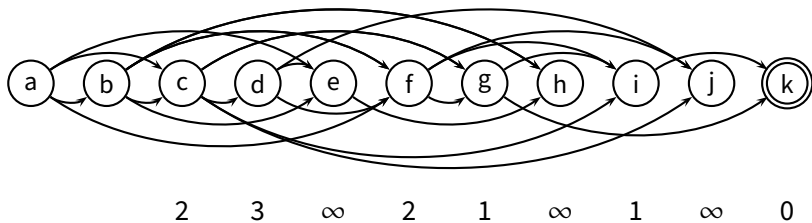
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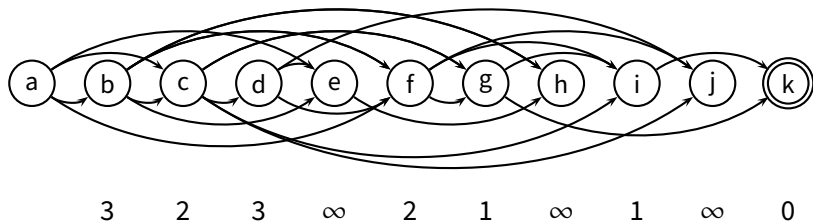
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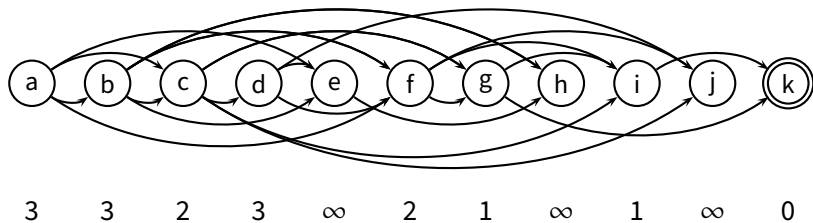
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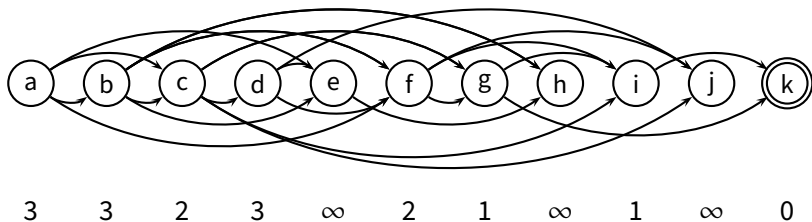
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- Since G is a DAG, computing D_y with $y \in \text{Adj}(x)$ can be considered a *subproblem* of computing D_x
 - ▶ we build the solution bottom-up, storing the subproblem solutions

Longest Increasing Subsequence

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- Given a sequence of numbers a_1, a_2, \dots, a_n , an *increasing subsequence* is any subset $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ such that $1 \leq i_1 < i_2 < \dots < i_k \leq n$, and such that

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A maximal-length subsequence is

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- Combining the subproblems

$$L(j) = 1 + \max\{L(i) \mid i < j \wedge a_i < a_j\}$$

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 - ▶ derive the solution from (one of) the solutions to the subproblems

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 - ▶ **exercise:** find a counter-example

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 - ▶ in dynamic programming, it is typical to reduce $L(j)$ into $L(j - 1)$
 - ▶ this is one reason why recursion does not work so well for dynamic programming
- Divide-and-conquer splits the problem into ***independent subproblems***
 - ▶ in dynamic programming, subproblems typically overlap
 - ▶ pretty much the same argument as above

Dynamic Programming vs. Greedy

- Greedy: requires the *greedy-choice property*
 - ▶ greedy: *greedy choice* plus *one subproblem*
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- Greedy: requires the ***greedy-choice property***
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 - ▶ greedy choice typically *before* proceeding to the subproblem
 - ▶ no need to store the result of each subproblem
- Dynamic programming: ***more general***
 - ▶ does not need the greedy-choice property
 - ▶ typically looks at several subproblems
 - ▶ “dynamically” choose one of them to obtain a global solution
 - ▶ typically works bottom-up
 - ▶ typically reuses solutions of the subproblems

- Prefix/suffix subproblems

- ▶ *Input:* x_1, x_2, \dots, x_n
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Z u r i g o

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ⇓ | | + | + | | - | - | |
| Z | | r | i | | | | |
| L | u | | | g | a | n | o |
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| Z | r | i | | | | | | | | | | | |
| L | u | | g | a | n | o | | L | u | g | a | n | o |
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| L | u | | | | g | a | n |
| | | | | | | | o |
| Z | u | r | | i | g | | o |

| | | | | | |
|---|---|---|---|---|---|
| | ↓ | | ↓ | ↓ | ↓ |
| Z | | r | | i | g |
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| | | | | | o |
| Z | u | r | | i | g |
| | | | | | o |

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- This suggests a way to combine the subproblems; let $diff(i, j) = 1$ iff $x[i] \neq y[j]$ or 0 otherwise

$$E(i, j) = \min\{1 + E(i - 1, j), \\ 1 + E(i, j - 1), \\ diff(i, j) + E(i - 1, j - 1)\}$$

■ Problem definition

- ▶ *Input:* a set of n objects with their weights w_1, w_2, \dots, w_n and their values v_1, v_2, \dots, v_n , and a maximum weight W
- ▶ *Output:* a subset K of the objects such that $\sum_{i \in K} w_i \leq W$ and such that $\sum_{i \in K} v_i$ is maximal

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■ Dynamic-programming solution

- ▶ let $K(w, j)$ be the maximum value achievable at maximum capacity w using the first j items (i.e., items $1 \dots j$)
- ▶ considering the j th element, we can either “use it or loose it,” so

$$K(w, j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}$$

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FIBONACCI(n)
```

```
1  if n == 0
```

```
2      return 0
```

```
3  elseif n == 1
```

```
4      return 1
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```
5  else return FIBONACCI(n - 1) + FIBONACCI(n - 2)
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- *Recursion solves the same problem over and over again*

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- **Idea:** “cache” the results

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- **Idea:** “cache” the results

```
FIBONACCI( $n$ )
1  if  $n == 0$ 
2      return 0
3  elseif  $n == 1$ 
4      return 1
5  elseif  $(n, x) \in H$  // a hash table  $H$  “caches” results
6      return  $x$ 
7  else  $x = \mathbf{FIBONACCI}(n - 1) + \mathbf{FIBONACCI}(n - 2)$ 
8      INSERT( $H, n, x$ )
9      return  $x$ 
```

- Idea also known as *memoization*

■ *Greedy*

1. start with the greedy choice
2. add the solution to the remaining subproblem

A nice tail-recursive algorithm

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3. in practice, solve the subproblems bottom-up

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 - ▶ Yes, because $2 + 134 + 78 = 214$
- **Puzzle 1:** is it possible to insert some '+' signs in the strings of digits to obtain the corresponding target number?

| <i>digits</i> | <i>target</i> |
|---|---------------|
| 646805736141599100791159198 | 472004 |
| 6152732017763987430884029264512187586207273294807 | 560351 |
| 48796142803774467559157928 | 326306 |
| 195961521219109124054410617072018922584281838218 | 7779515 |