

Elementary Data Structures and Hash Tables

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- Common concepts and notation
- Stacks
- Queues
- Linked lists
- Trees
- Direct-access tables
- Hash tables

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 - ▶ e.g., a *heap* needs an array A to store the keys, plus a variable $A.heap-size$ to remember how many elements are in the heap

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 - ▶ **STACK-EMPTY**(S) returns TRUE if and only if S is empty
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- *Implementation*
 - ▶ using an array
 - ▶ using a linked list
 - ▶ ...

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3  else return FALSE
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1  if  $S.top == 0$ 
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PUSH(S,x)

```
1   $S.top = S.top + 1$ 
2   $S[S.top] = x$ 
```

POP(S)

```
1  if STACK-EMPTY( $S$ )
2      error "underflow"
3  else  $S.top = S.top - 1$ 
4      return  $S[S.top + 1]$ 
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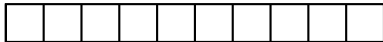
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- *Implementation*

- ▶ Q is an array of fixed length $Q.length$
 - ▶ i.e., Q holds at most $Q.length$ elements
 - ▶ enqueueing more than Q elements causes an “overflow” error
- ▶ $Q.head$ is the position of the “head” of the queue
- ▶ $Q.tail$ is the first empty position at the tail of the queue

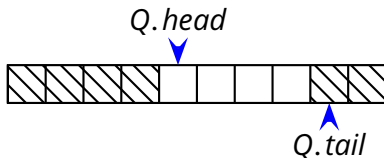
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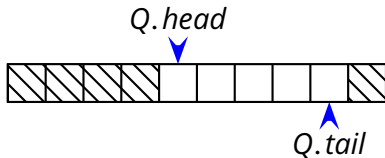
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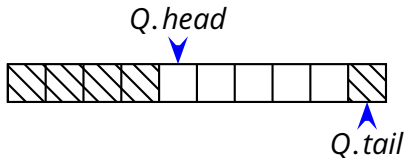
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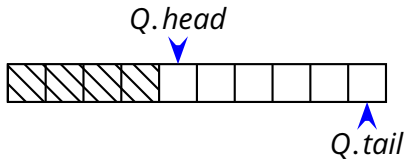
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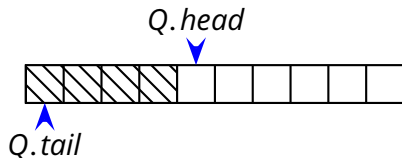
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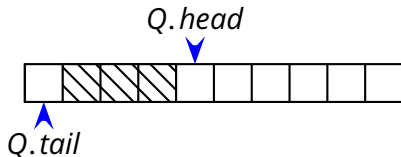
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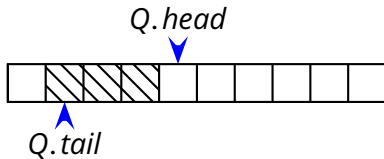
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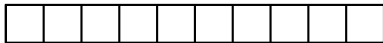
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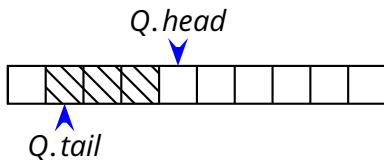
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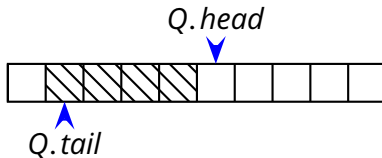
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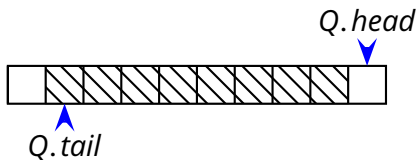
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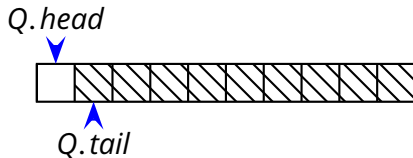
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■ *Interface*

- ▶ **LIST-INSERT**(L, x) adds element x at beginning of a list L
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■ *Implementation*

- ▶ a *doubly-linked* list
- ▶ each element x has two “links” $x.prev$ and $x.next$ to the previous and next elements, respectively
- ▶ each element x holds a key $x.key$
- ▶ it is convenient to have a dummy “sentinel” element $L.nil$

Linked List With a "Sentinel"

LIST-INIT(*L*)

- 1 $L.nil.prev = L.nil$
- 2 $L.nil.next = L.nil$

LIST-INSERT(*L*, *x*)

- 1 $x.next = L.nil.next$
- 2 $L.nil.next.prev = x$
- 3 $L.nil.next = x$
- 4 $x.prev = L.nil$

LIST-SEARCH(*L*, *k*)

- 1 $x = L.nil.next$
- 2 **while** $x \neq L.nil \wedge x.key \neq k$
- 3 $x = x.next$
- 4 **return** x

■ *Structure*

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■ *Implementation*

- ▶ for each node $x \neq T.root$, $x.parent$ is x 's parent node
- ▶ fixed branching:
e.g., $x.left-child$ and $x.right-child$ in a *binary tree*
- ▶ unbounded branching:
 $x.left-child$ is x 's first (leftmost) child
 $x.right-sibling$ is x closest sibling to the right

<i>Algorithm</i>	<i>Complexity</i>
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STACK-EMPTY	
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LIST-SEARCH	$\Theta(n)$

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 - ▶ ***hash tables***

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DIRECT-ADDRESS-INSERT(T, k)

1 $T[k] = \text{TRUE}$

DIRECT-ADDRESS-DELETE(T, k)

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DIRECT-ADDRESS-SEARCH(T, k)

1 **return** $T[k]$

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■ The **space complexity** is $\Theta(|U|)$

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- *Can we have the benefits of a direct-address table but with a table of reasonable size?*

■ *Idea*

- ▶ use a table T with $|T| \ll |U|$
- ▶ map each key $k \in U$ to a position in T , using a ***hash function***

$$h : U \rightarrow \{1, \dots, |T|\}$$

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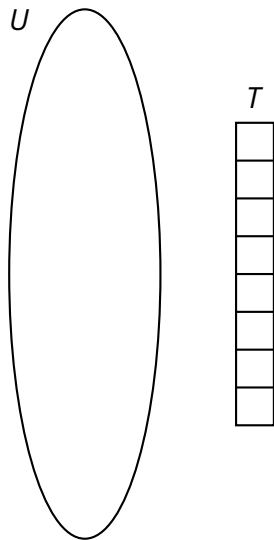
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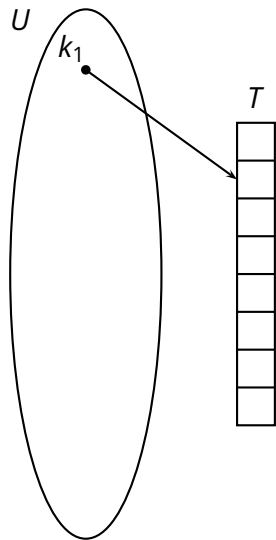
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What if two distinct keys $k_1 \neq k_2$ collide? (i.e., $h(k_1) = h(k_2)$)

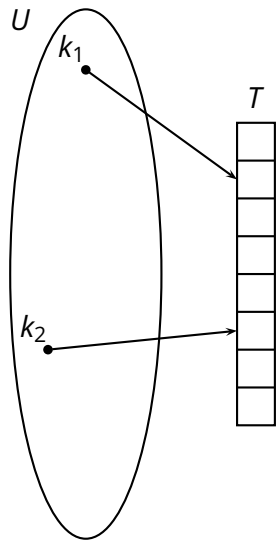
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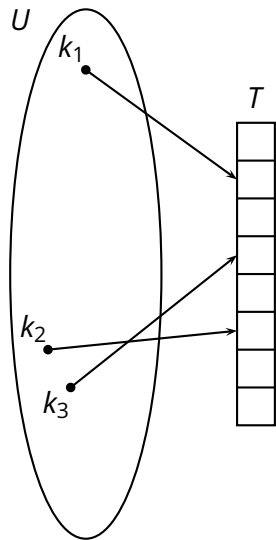
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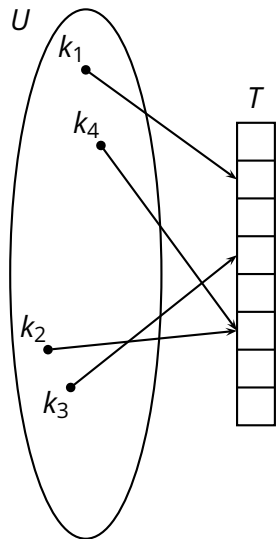
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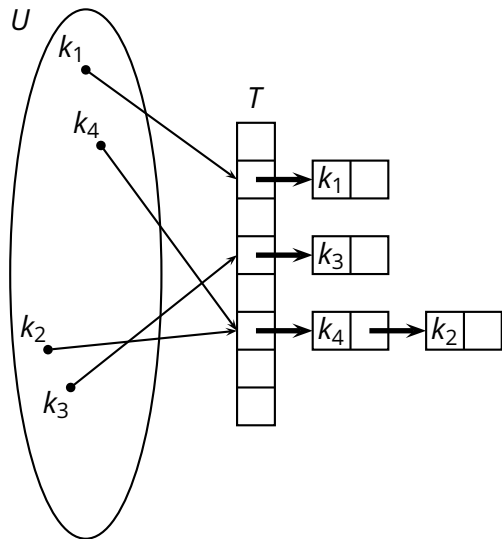
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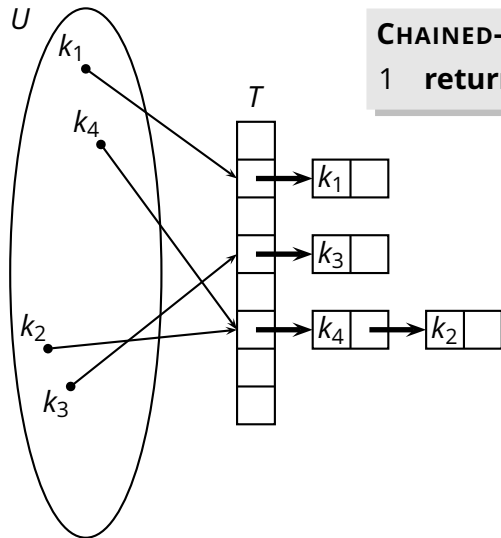


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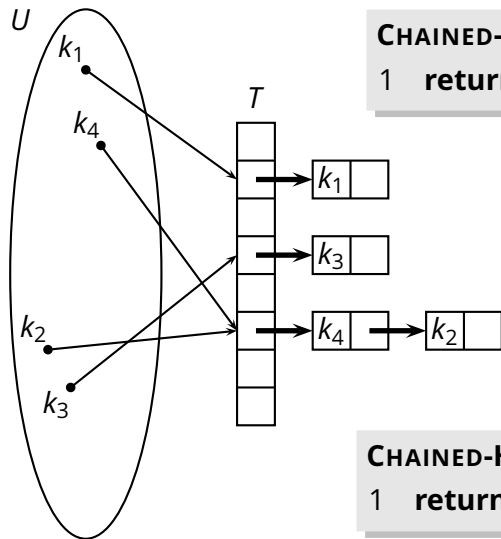




CHAINED-HASH-INSERT(T, k)

1 **return** LIST-INSERT($T[h(k)], k$)

Hash Table



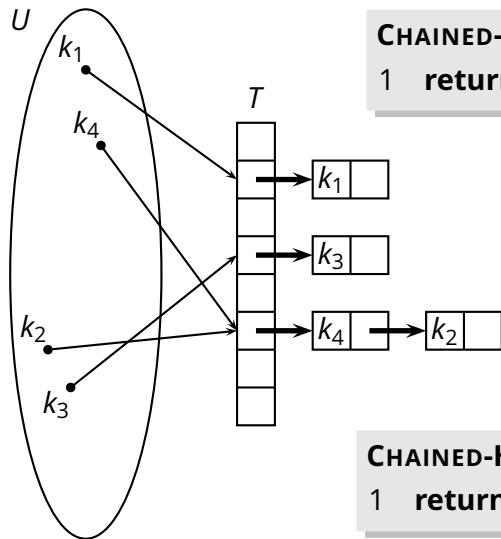
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load factor

$$\alpha = \frac{n}{|T|}$$

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- We further assume that $h(k)$ can be computed in $O(1)$ time

- We assume **uniform hashing** for our hash function $h : U \rightarrow \{1 \dots |T|\}$ (where $|T| = T.length$)

$$\Pr[h(k) = i] = \frac{1}{|T|} \quad \text{for all } i \in \{1 \dots |T|\}$$

(The formalism is actually a bit more complicated.)

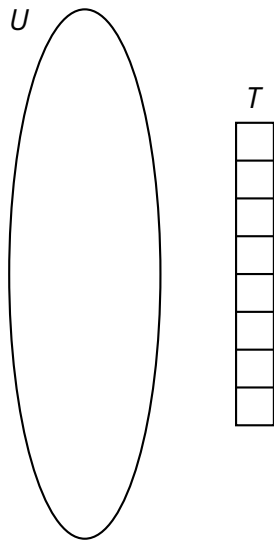
- So, given n distinct keys, the expected length n_i of the linked list at position i is

$$E[n_i] = \frac{n}{|T|} = \alpha$$

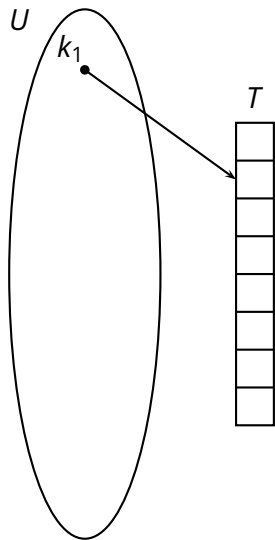
- We further assume that $h(k)$ can be computed in $O(1)$ time
- Therefore, the complexity of **CHAINED-HASH-SEARCH** is

$$\Theta(1 + \alpha)$$

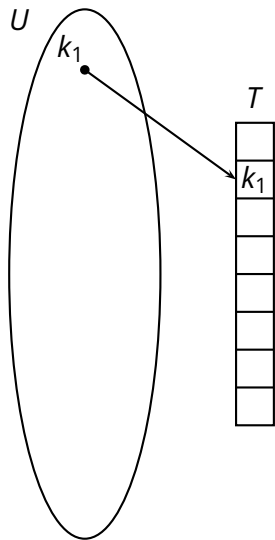
Open-Address Hash Table



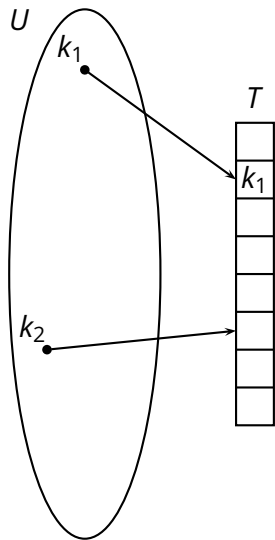
Open-Address Hash Table



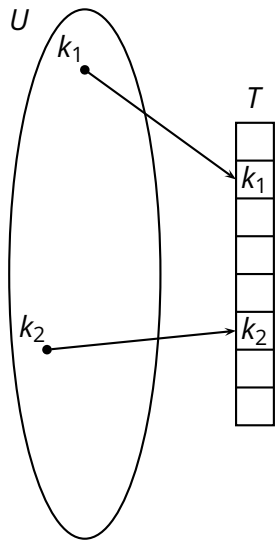
Open-Address Hash Table



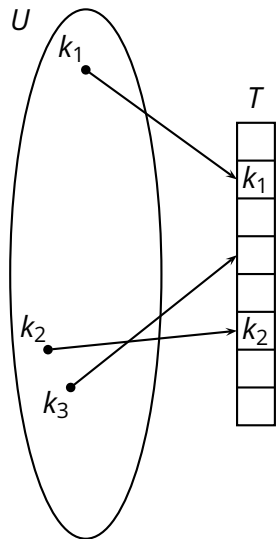
Open-Address Hash Table



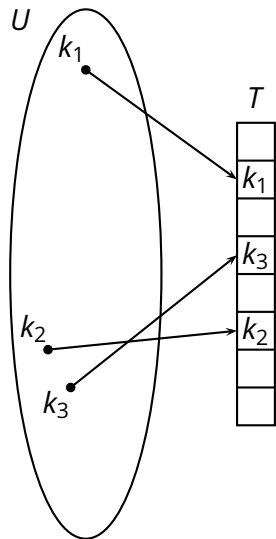
Open-Address Hash Table



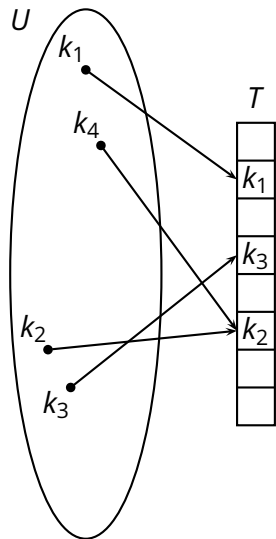
Open-Address Hash Table



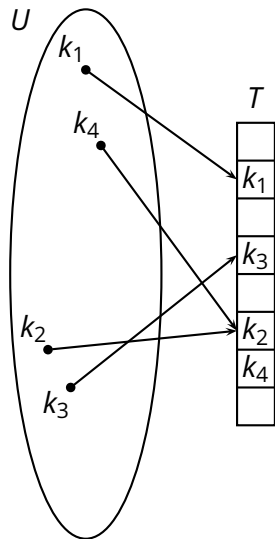
Open-Address Hash Table



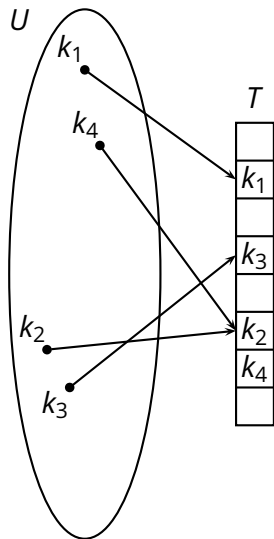
Open-Address Hash Table



Open-Address Hash Table



Open-Address Hash Table



HASH-INSERT(T, k)

```
1  $j = h(k)$ 
2 for  $i = 1$  to  $T.length$ 
3     if  $T[j] == \text{NIL}$ 
4          $T[j] = k$ 
5         return  $j$ 
6     elseif  $j < T.length$ 
7          $j = j + 1$ 
8     else  $j = 1$ 
9     error "overflow"
```

Open-Addressing (2)

- *Idea:* instead of using linked lists, we can store all the elements in the table
 - ▶ this implies $\alpha \leq 1$

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- *Idea:* instead of using linked lists, we can store all the elements in the table
 - ▶ this implies $\alpha \leq 1$
- When a collision occurs, we simply find another free cell in T
- A sequential “probe” may not be optimal
 - ▶ can you figure out why?

Open-Addressing (3)

```
HASH-INSERT( $T, k$ )  
1  for  $i = 1$  to  $T.length$   
2   $j = h(k, i)$   
3      if  $T[j] == \text{NIL}$   
4           $T[j] = k$   
5          return  $j$   
6  error "overflow"
```

```
HASH-INSERT( $T, k$ )  
1  for  $i = 1$  to  $T.length$   
2   $j = h(k, i)$   
3      if  $T[j] == \text{NIL}$   
4           $T[j] = k$   
5          return  $j$   
6  error "overflow"
```

■ Notice that $h(k, \cdot)$ must be a **permutation**

- ▶ i.e., $h(k, 1), h(k, 2), \dots, h(k, |T|)$ must cover the entire table T