Divide-and-Conquer Algorithms

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Outline

- Merging (or set union)
- Searching
- Sorting
- Multiplying
- Computing the *median*

Merging (Set Union)

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- *Input:* sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$

  Output: a sequence (a set) $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

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- Example:
  $A = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10 \rangle$
  $B = \langle 51, 21, 14, 15, 27, 31, 2 \rangle$

  $X =$

Merging (Set Union)

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A Simple Merge Algorithm

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  - iterate through every position i, first through A, and then B
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  - output b_i if b_i is not in $\langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_{i-1} \rangle$

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```
MergeSimple(A, B)
for i = 1 to length(A)
    if not Find(A[1 .. i − 1], A[i])
        output A[i]
for i = 1 to length(B)
    if not Find(A, B[i]) and not Find(B[1 .. i − 1], B[i])
        output B[i]
```

Complexity

```
MERGESIMPLE(A, B)
for i = 1 to length(A)
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```

let $n = length(A) + length(B)$

$$T(n) = \sum_{i=1}^{length(A)} T_{\text{FIND}}(i) + \sum_{i=1}^{length(B)} \left(T_{\text{FIND}}(i) + T_{\text{FIND}}(length(A))\right)$$

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Searching

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  Output: TRUE if *A* contains *key*, or FALSE otherwise

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```
FindInList(A, key)
item = first(A)
while item ≠ last(A)
        if value(item) == key
                return TRUE
        item = next(item)
return FALSE
```

Searching on a List

- *Input:* a sequence A and a value *key*
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Searching on a List

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Complexity of MergeSimple

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Searching (2)

- *Input:* a *sorted* sequence A and a value *key*
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- *Input:* a *sorted* sequence *A* and a value *key*
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BinarySearch(A, key)
first = 1
last = length(A)
while first ≤ last
      middle = ⌈(first + last)/2⌉
      if A[middle] == key
            return TRUE
      elseif first = last
            return FALSE
      elseif A[middle] > key
            last = middle − 1
      else first = middle + 1
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```

Binary Search

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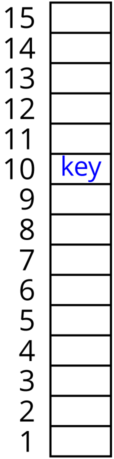

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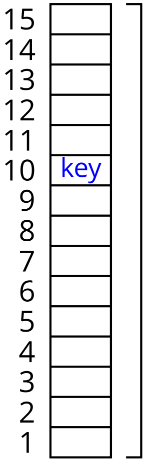

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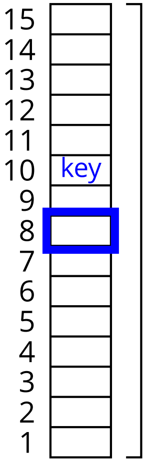

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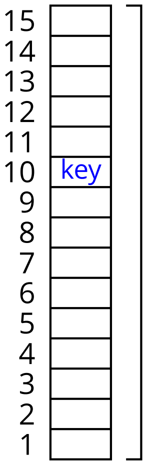

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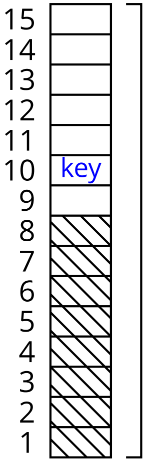

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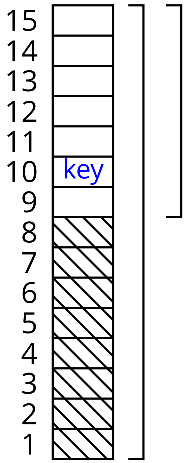

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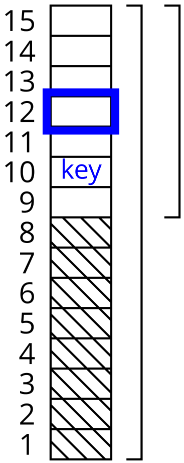

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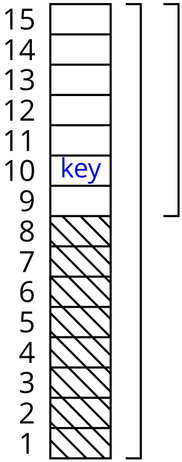

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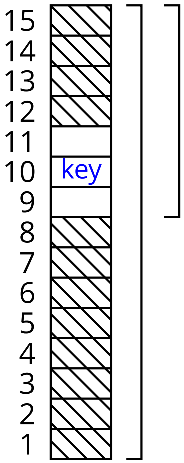

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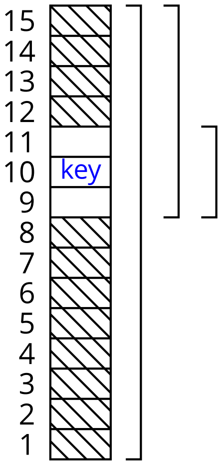

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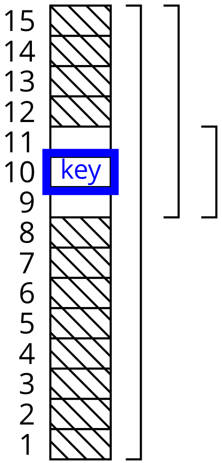

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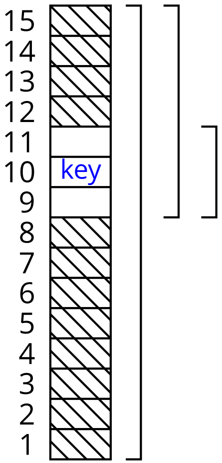

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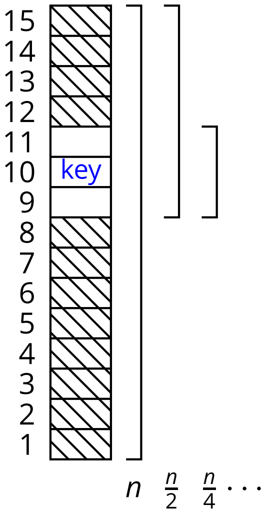

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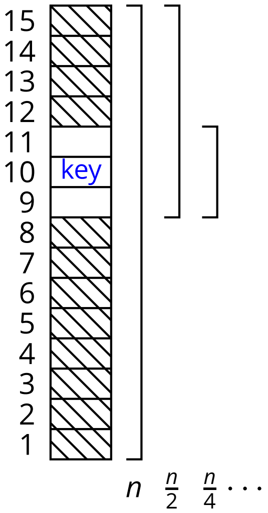

$T(n) = O(\log n)$

Merging Sorted Sequences

- A slightly different problem:

  Input: two *sorted* sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$, where $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_m$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

  - every element of A appears once in X
  - every element of B appears once in X
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```
MERGESIMPLE2(A, B)
for i = 1 to length(A)
    if not BINARYSEARCH(A[1 . . i − 1], A[i])
        output A[i]
for i = 1 to length(B)
    if not BINARYSEARCH(A, B[i])
    and not BINARYSEARCH(B[1 . . i − 1], B[i])
        output B[i]
```

A Better Merge Algorithm

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```

$$T(n) = \sum_{i=1}^{n} O(\log i) = O(n \log n)$$

Better than $O(n^2)$, but can we do even better than $O(n \log n)$?

An Even Better Merge Algorithm

- *Intuition:* A and B are *sorted*
  e.g.
  $A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$
  $B = \langle 1, 5, 6, 7, 34, 35, 40, 41, 43 \rangle$

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  so just like in **BinarySearch** I can avoid looking for an element x if the *first* element I see is $y > x$

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  so just like in **BINARYSEARCH** I can avoid looking for an element x if the *first* element I see is $y > x$

- High-level algorithm strategy
  - step through every position i of A and every position j of B
  - output a_i and advance i if $a_i \leq b_j$ or if j is beyond the end of B
  - output b_j and advance j if $a_i \geq b_j$ or if i is beyond the end of A

A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

MERGE Algorithm

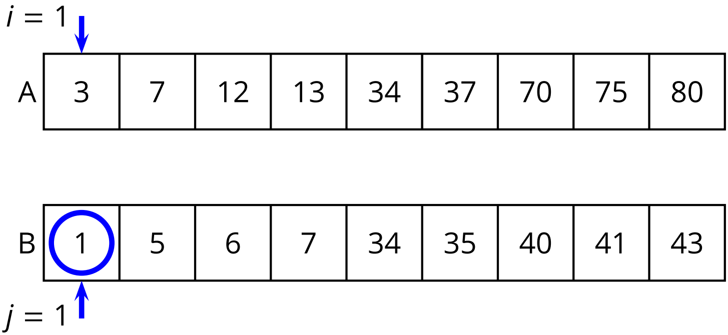
$i = 1$

A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

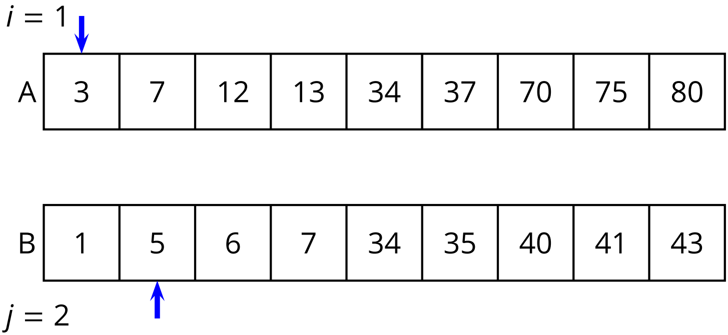
$j = 1$

Output:



Output:

MERGE Algorithm



Output: 1

MERGE Algorithm

i = 1

A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

j = 2

Output: 1

MERGE Algorithm

$i = 2$

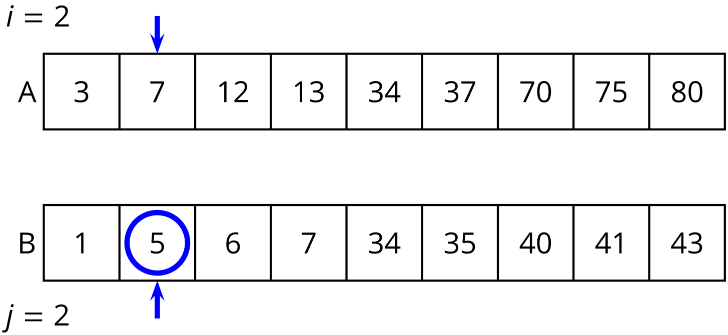
A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 2$

Output: 1 3

MERGE Algorithm



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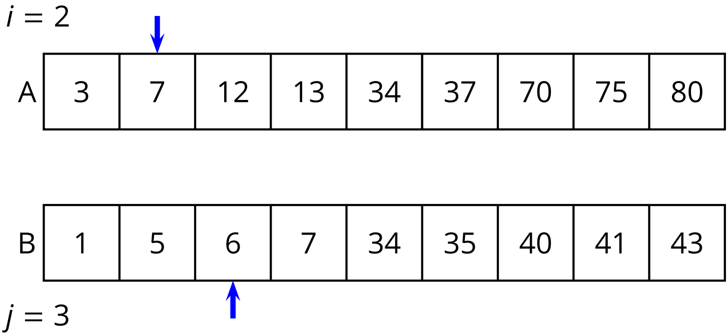

Output: 1 3 5

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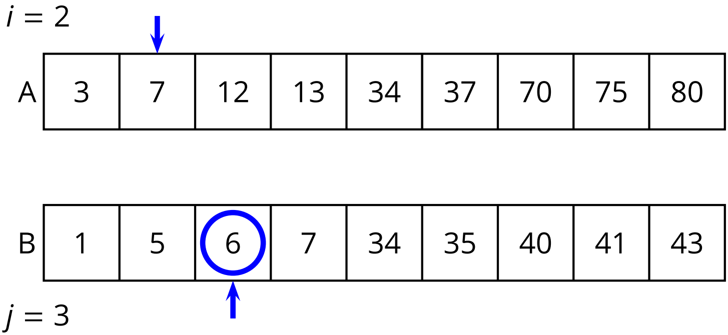

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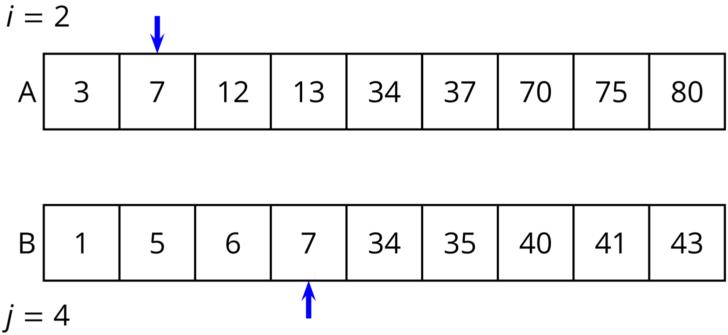

Output: 1 3 5 6

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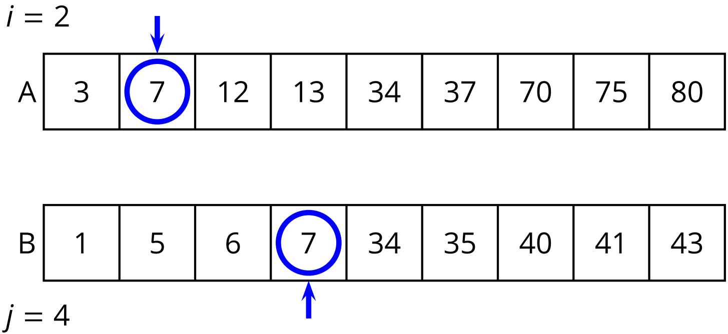

Output: 1 3 5 6

$i = 3$

A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 5$

Output: 1 3 5 6 7

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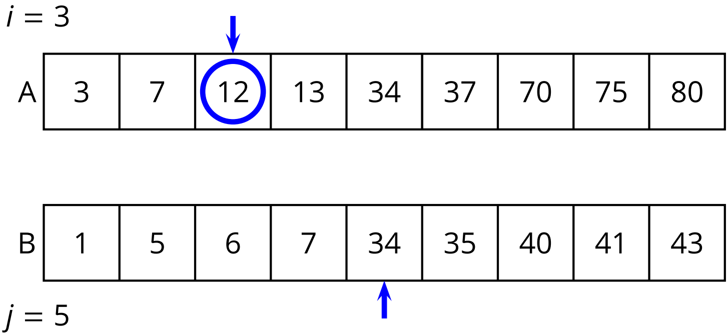

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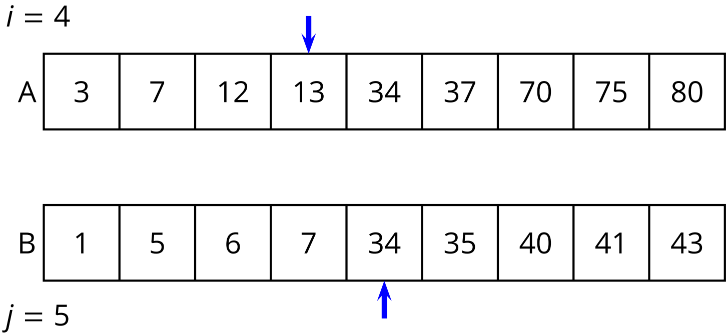

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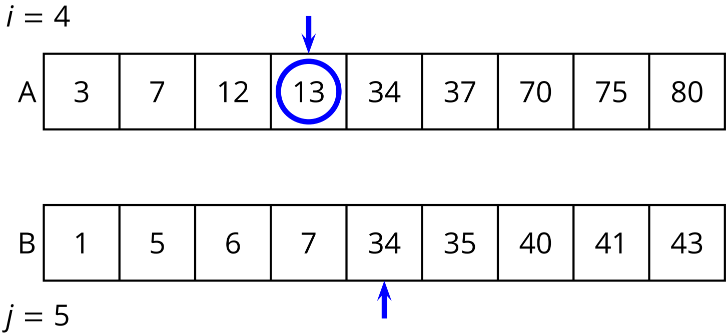

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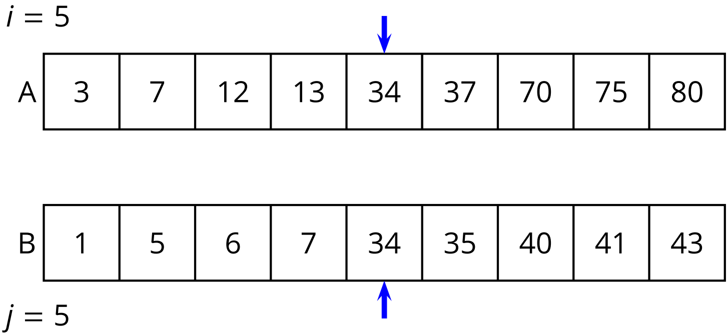

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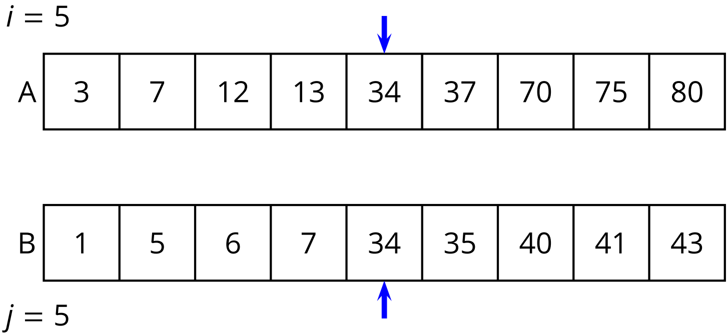

Output: 1 3 5 6 7 12 13. . .

MERGE Algorithm (2)

```
MERGE(A, B)
i, j = 1
X = ∅
while i ≤ length(A) or j ≤ length(B)
        if i > length(A)
                X = X ∘ B[j]            // appends B[j] to X
                j = j + 1
        elseif j > length(B)
                X = X ∘ A[i]
                i = i + 1
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- This algorithm is incorrect! (Exercise: fix it)

Complexity of MERGE

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$$T(n) = \Theta(n)$$

- Can we do better? No!
  - we have to output $n = \mathit{length}(A) + \mathit{length}(B)$ elements

Using MERGE

- So now we have a *linear-complexity* merge procedure
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  - this suggests a recursive algorithm

Merge Sort

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MERGESORT(A)
if length(A) == 1
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m = ⌊length(A)/2⌋
A_L = MERGESORT(A[1 . . m])
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return MERGE(A_L, A_R)
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- *General strategy:* given a problem P on input data A
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- Complexity analysis

$$T(n) = T_{\text{divide}} + \sum_{i=1}^{k} T(|A_i|) + T_{\text{combine}}$$

we will analyze this formula another time…

A Divide-and-Conquer Merge

```
MERGER(A, B)
if length(A) == 0
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if A[1] < B[1]
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- Can we do better?

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- Can we do better? No! (We knew that already)

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$x = \boxed{X_L}\,\boxed{X_R}$ and $y = \boxed{Y_L}\,\boxed{Y_R}$

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  $x = \boxed{X_L}\boxed{X_R}$ and $y = \boxed{Y_L}\boxed{Y_R}$

  which means $x = 2^{\ell/2}x_L + x_R$ and $y = 2^{\ell/2}y_L + y_R$, so...

$$\begin{aligned} xy &= (2^{\ell/2}x_L + x_R)(2^{\ell/2}y_L + y_R) \\ &= 2^{\ell}x_L y_L + 2^{\ell/2}(x_L y_R + x_R y_L) + x_R y_R \end{aligned}$$

  we reduced the problem of multiplying two numbers of ℓ bits into the problem of multiplying *four* numbers of $\ell/2$ bits...

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  Only 3 multiplications: x_Ly_L, $(x_L + x_R)(y_R + y_L)$, and x_Ry_R

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$$T(\ell) = 3T(\ell/2) + O(\ell)$$

  which, as we will see, leads to a much better complexity

$$T(\ell) = O(\ell^{\log_2 3}) = O(\ell^{1.59})$$

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- How long does it take? $T(n) = T_{\text{MERGESORT}}(n) = O(n \log n)$
- Can we do better? Let's try *divide-and-conquer*...

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    the 6th smallest element of A—a.k.a. $select(A, 6)$—is 8

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It is the 2nd smallest value of A_R

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We use $select(A, k)$ to denote the k-smallest element of A

$$select(A, k) = \begin{cases} select(A_L, k) & \text{if } k \leq |A_L| \\ v & \text{if } |A_L| < k \leq |A_L| + |A_v| \\ select(A_R, k - |A_L| - |A_v|) & \text{if } k > |A_L| + |A_v| \end{cases}$$

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- We pick *a random element of A*

```
SELECTION(A, k)
v = A[random(1 ... |A|)]
A_L, A_v, A_R = ∅
for i = 1 to |A|
        if A[i] < v
                A_L = A_L ∪ A[i]
        elseif A[i] == v
                A_v = A_v ∪ A[i]
        else A_R = A_R ∪ A[i]
if k ≤ |A_L|
        return SELECTION(A_L, k)
elseif k > |A_L| + |A_v|
        return SELECTION(A_R, k − |A_L| − |A_v|)
else return v
```