More on Sorting: Quick Sort and Heap Sort

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Outline

- Another divide-and-conquer sorting algorithm
- The *heap*
- Heap sort

Sorting Algorithms Seen So Far

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Algorithm	Complexity			In place?
	worst	*average*	*best*	

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Merge-Sort	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no

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MERGE-SORT	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no
??		$\Theta(n \log n)$		***yes***
??	$\Theta(n \log n)$			***yes***

Using the Partitioning Algorithm

- *Basic step:* partition A in three parts based on a *chosen value* $v \in A$
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- *Can we partition A* ***in place****?*

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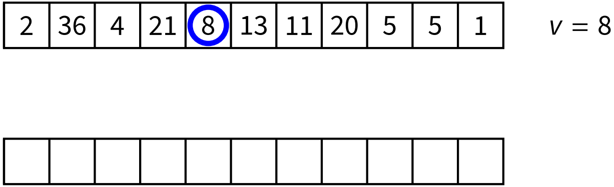

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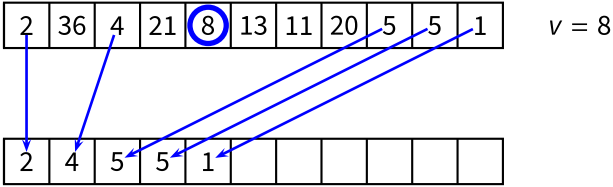

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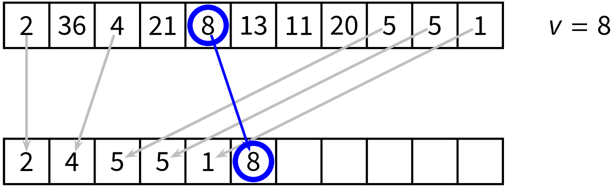

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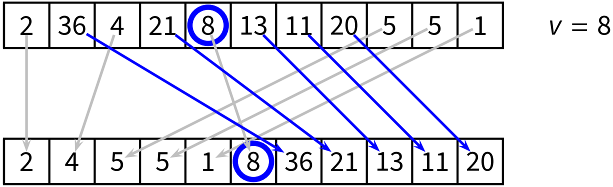

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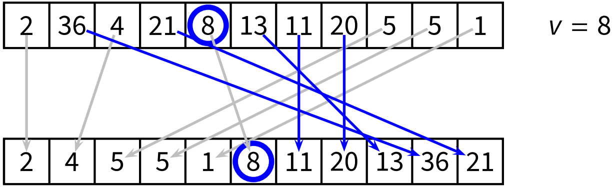

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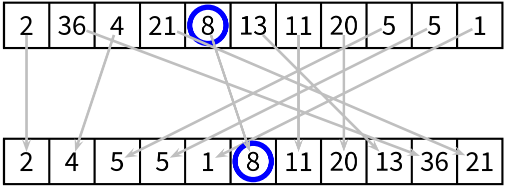

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$q = 6$

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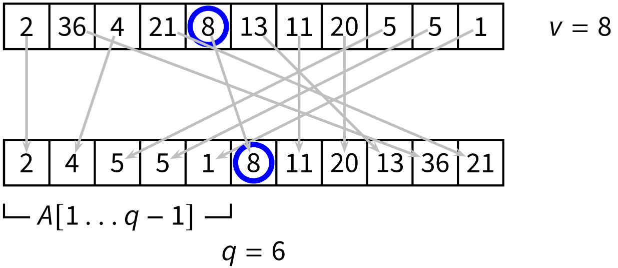

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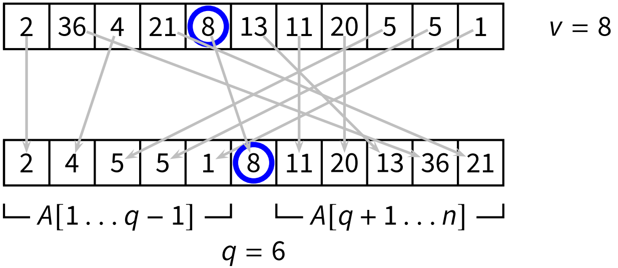

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```
QuickSort(A, begin, end)
if begin < end
        q = Partition(A, begin, end)
        QuickSort(A, begin, q − 1)
        QuickSort(A, q + 1, end)
```

Partition

- Start with $q = 1$
  - i.e., *assume all elements are greater than the pivot*

- Scan the array left-to-right, starting at position 2

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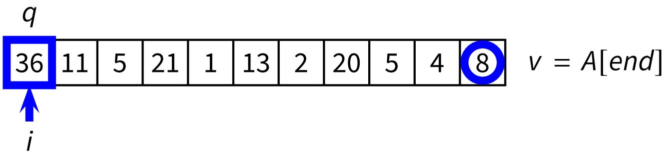

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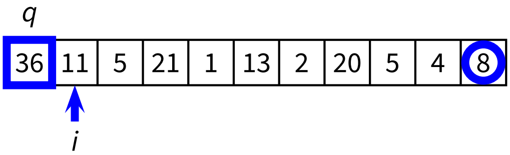

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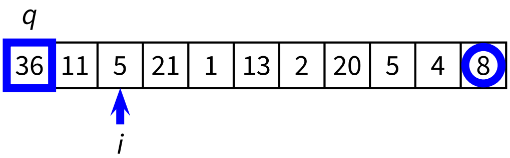

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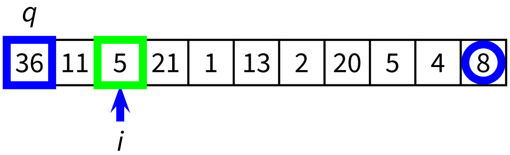

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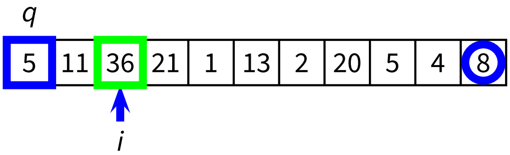

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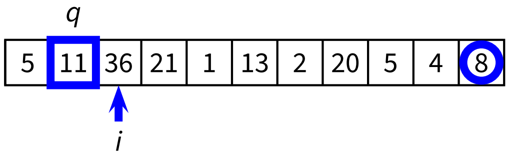

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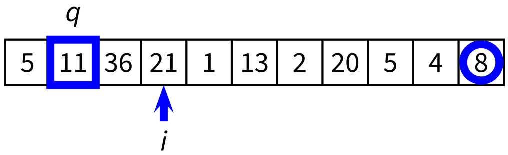

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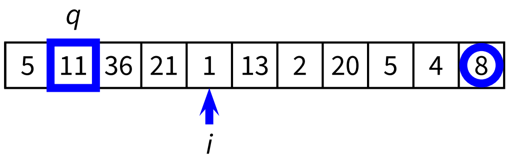

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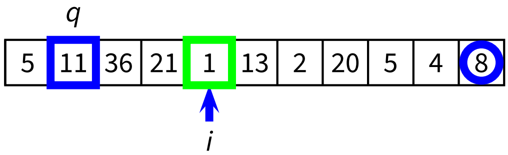

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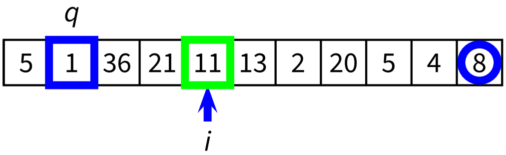

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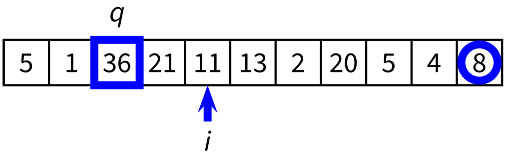

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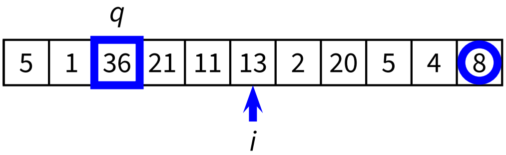

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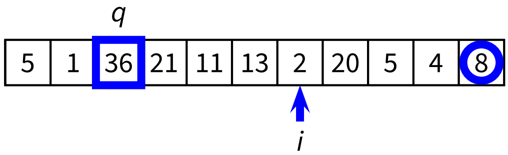

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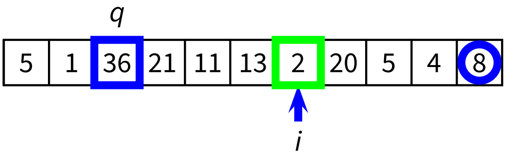

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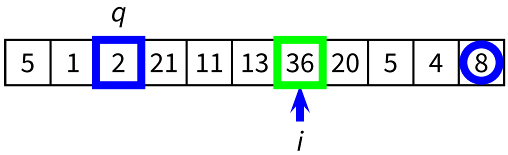

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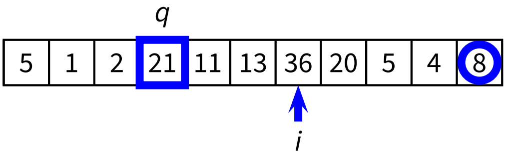

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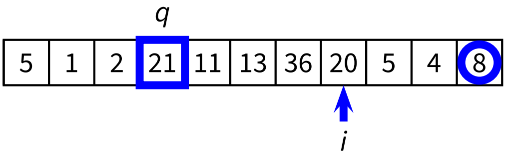

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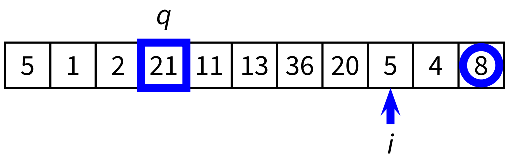

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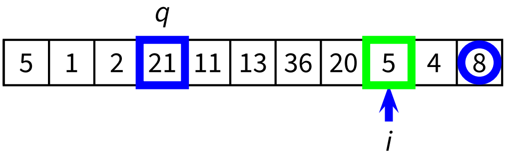

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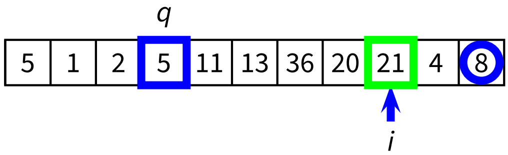

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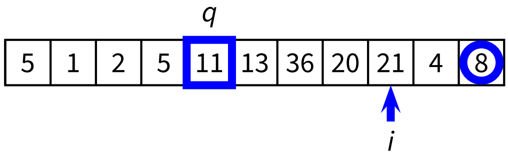

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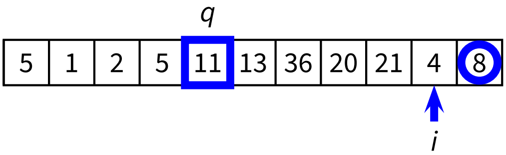

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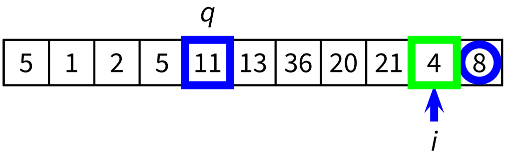

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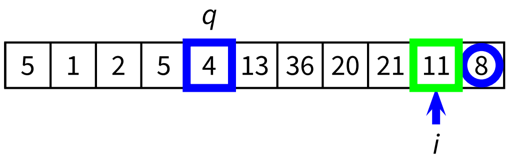

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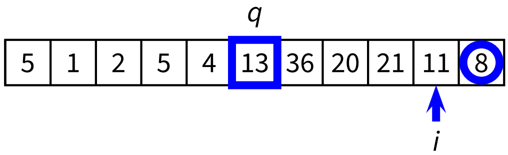

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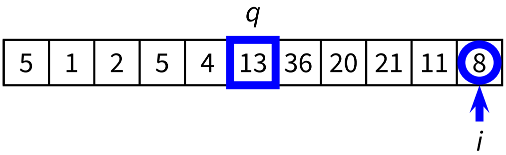

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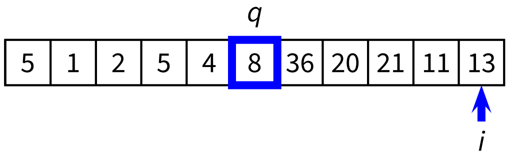

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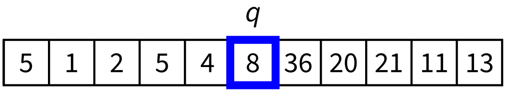

Complete QUICKSORT Algorithm

```
PARTITION(A, begin, end)
q = begin
v = A[end]
for i = begin to end
    if A[i] ≤ v
        swap A[i] and A[q]
        q = q + 1
return q − 1
```

```
QUICKSORT(A, begin, end)
if begin < end
    q = PARTITION(A, begin, end)
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$$T(n) = \Theta(n)$$

Complexity of QuickSort

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Sorting Algorithms Seen So Far

Algorithm	Complexity			In place?
	worst	*average*	*best*	
Insertion-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	yes
Selection-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	yes
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QuickSort				

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MERGE-SORT	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no
QUICKSORT	$\Theta(n^2)$	$\Theta(n \log n)$	$\Theta(n \log n)$	yes
??	$\Theta(n \log n)$			***yes***

- Our first real ***data structure***

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Binary Heap

- Our first real ***data structure***

- Interface
  - **BUILD-MAX-HEAP**(A) rearranges A into a max-heap
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Binary Heap: Structure

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- Conceptually a full binary tree

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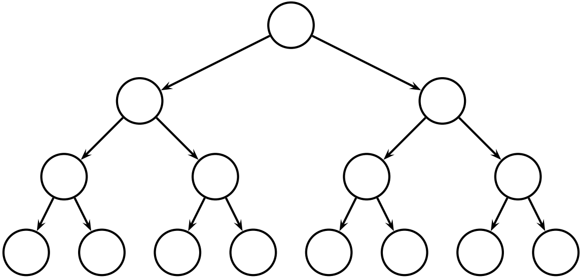

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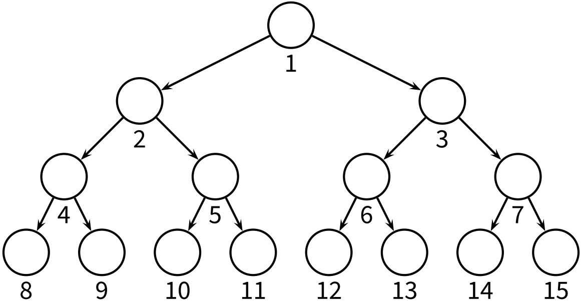

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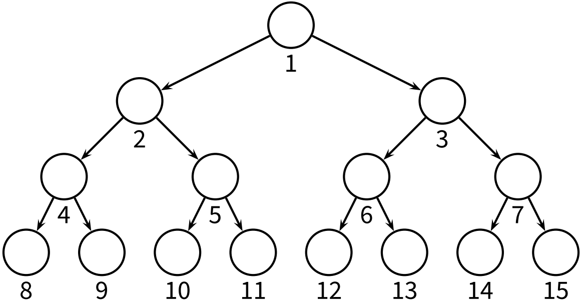

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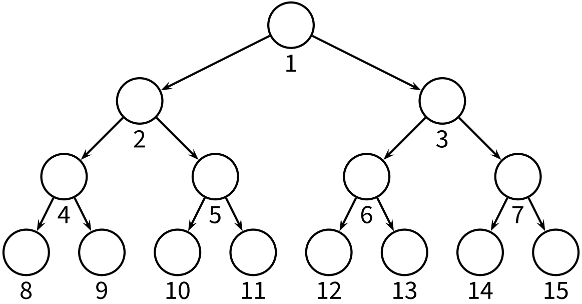

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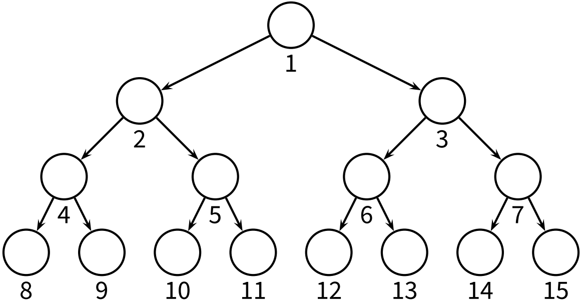

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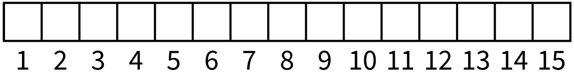

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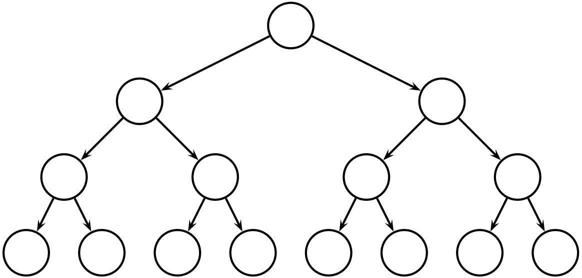

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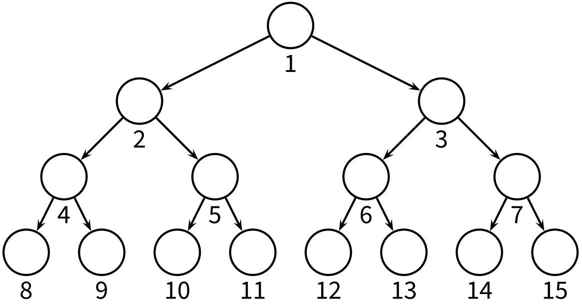

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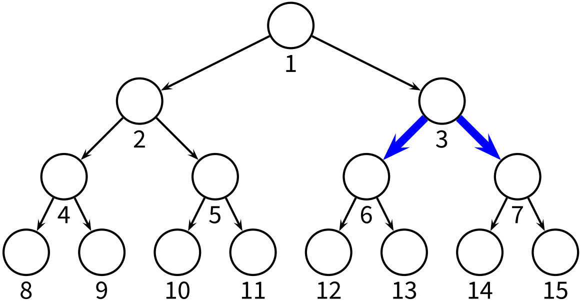

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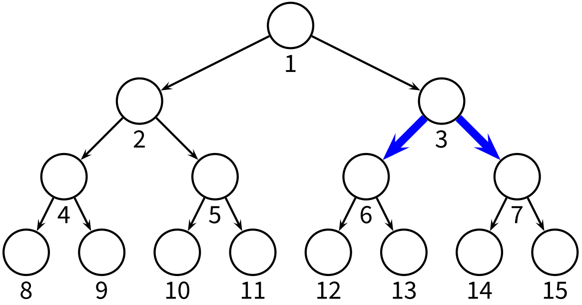


Parent(i)

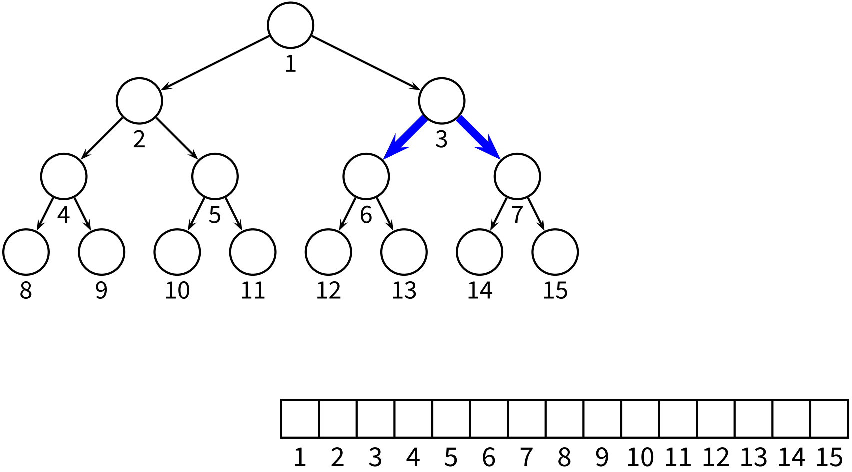
return $\lfloor i/2 \rfloor$

Left(i)

return $2i$

Right(i)

return $2i + 1$



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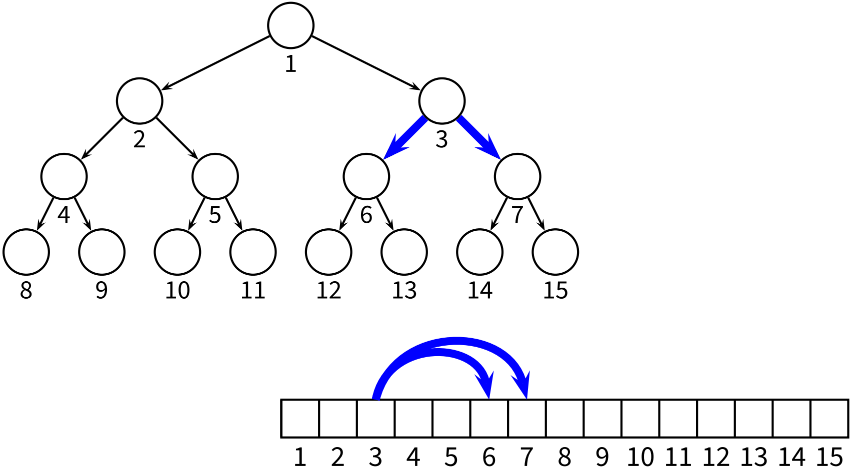
Binary Heap: Properties



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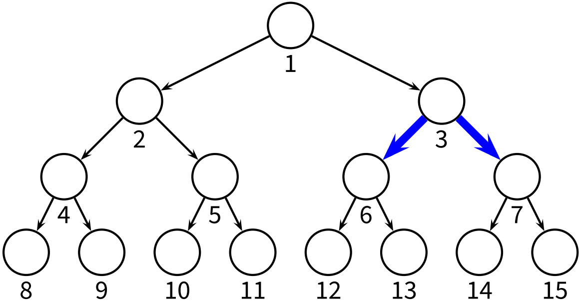

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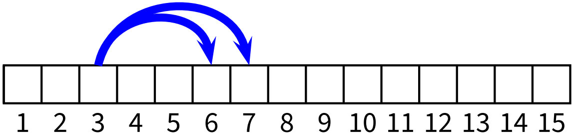

- ***Max-heap property:*** for all $i > 1$ $A[\text{PARENT}(i)] \geq A[i]$

Example

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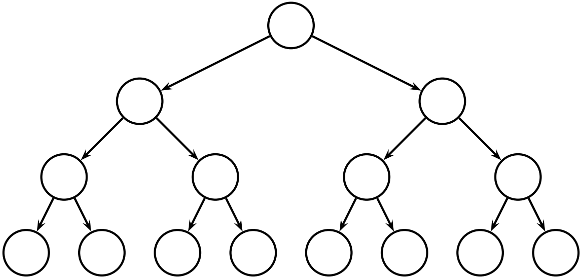

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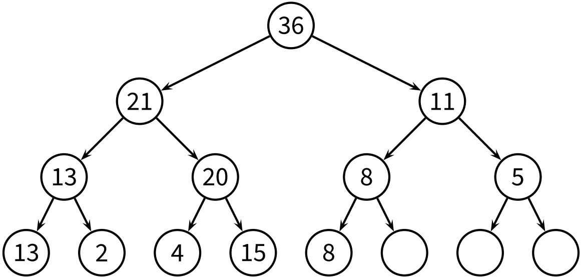

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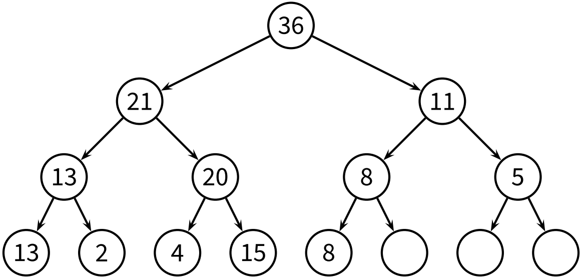

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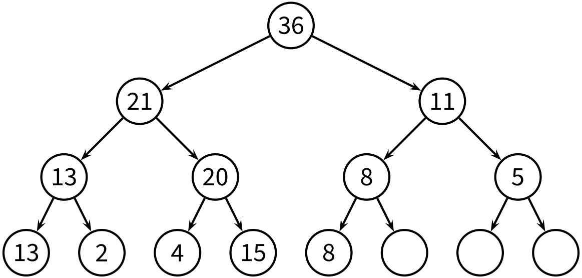

- Where is the max element?

- How can we implement **Heap-Extract-Max**?

Heap-Extract-Max

- **Heap-Extract-Max** procedure
  - extract the max key
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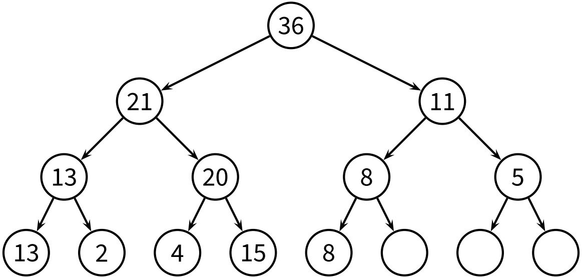

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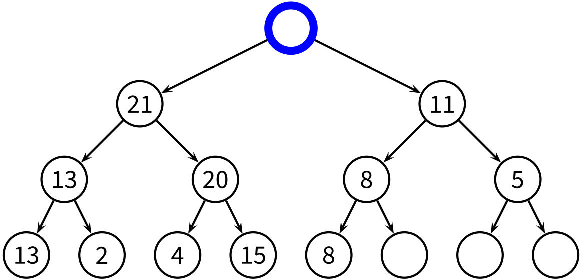

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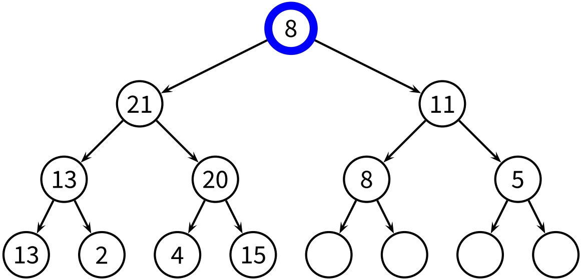

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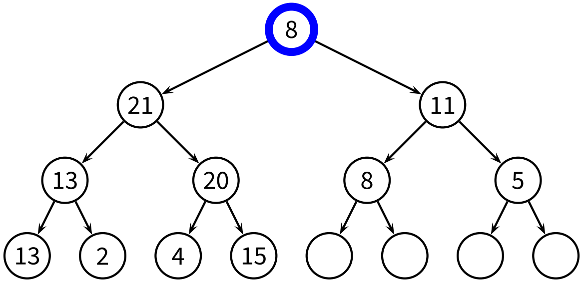

- Now we have two subtrees where the *max-heap property* holds

- **MAX-HEAPIFY**(A, i) procedure
  - *assume:* the *max-heap property* holds in the subtrees of node i
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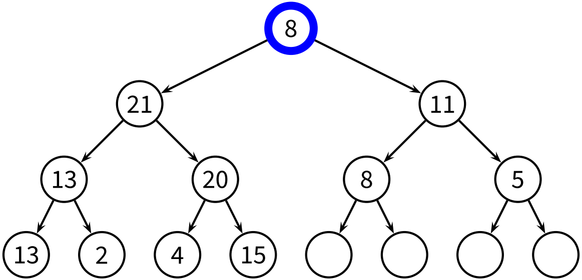

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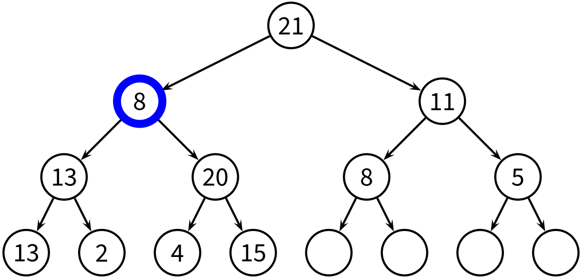

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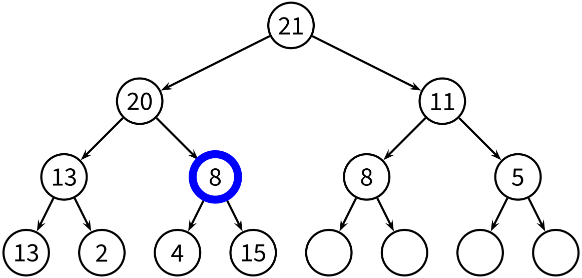

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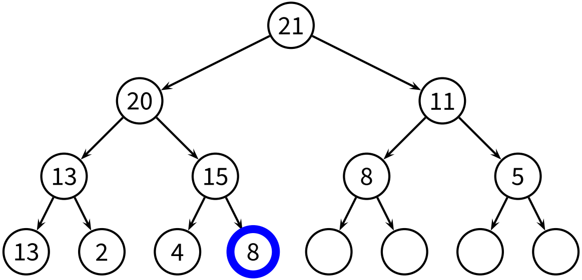

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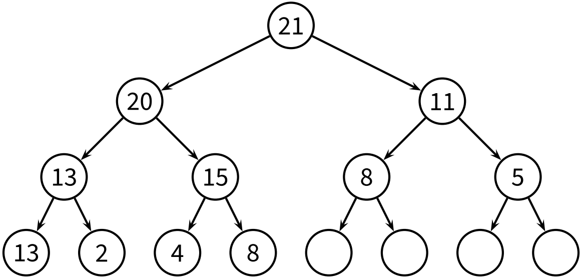

Max-Heapify

```
MAX-HEAPIFY(A, i)
l = LEFT(i)
r = RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
        largest = l
else largest = i
if r ≤ A.heap-size and A[r] > A[largest]
        largest = r
if largest ≠ i
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$$T(n) = \Theta(\log n)$$

Building a Heap

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BUILD-MAX-HEAP(A)
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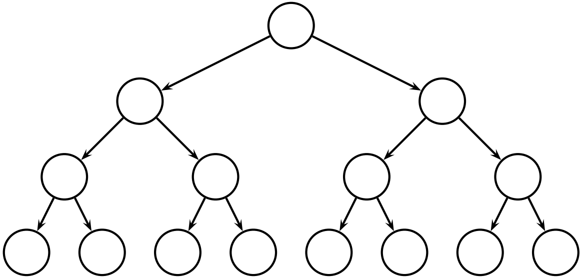

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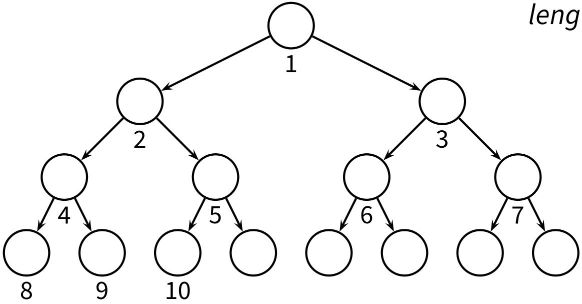

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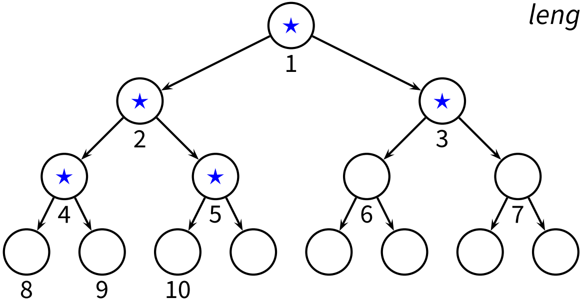

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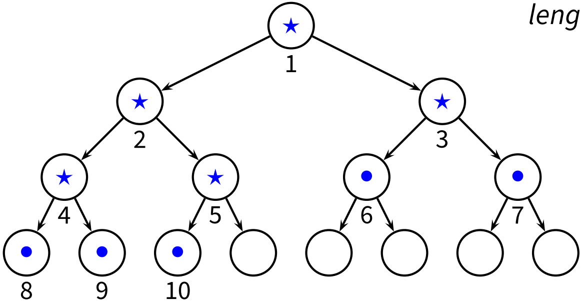

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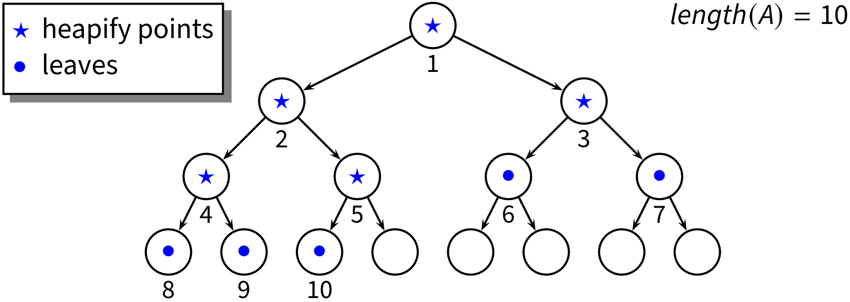

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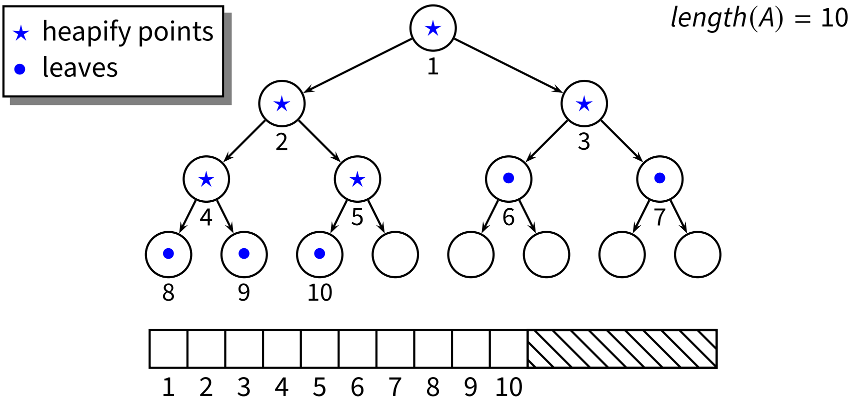

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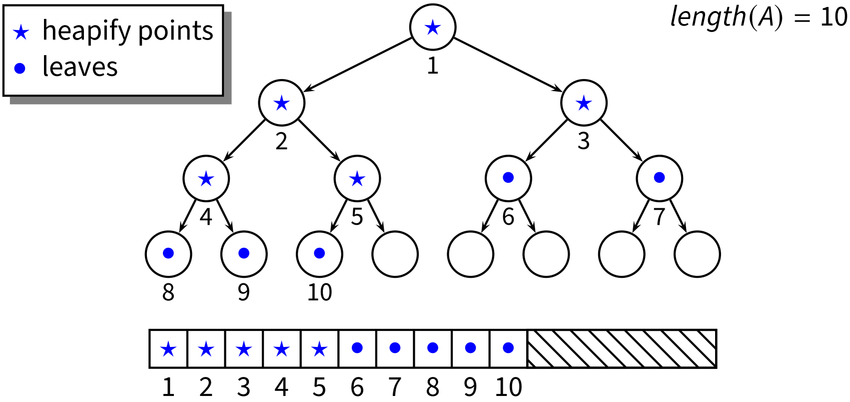

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HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i = length(A) downto 1
    swap A[i] and A[1]
    A.heap-size = A.heap-size − 1
    MAX-HEAPIFY(A, 1)
```

Heap Sort

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- What is the complexity of **HEAP-SORT**?

$$T(n) = \Theta(n \log n)$$

- Benefits
  - in-place sorting; worst-case is $\Theta(n \log n)$

Summary of Sorting Algorithms

Algorithm	Complexity			In place?
	worst	*average*	*best*	
INSERTION-SORT				

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MERGE-SORT	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no
QUICK-SORT				

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