

Analysis of Insertion Sort

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- Sorting
- Insertion Sort
- Analysis

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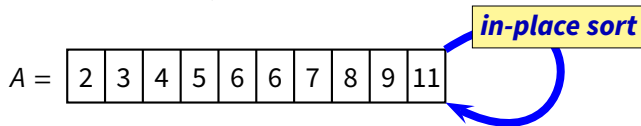
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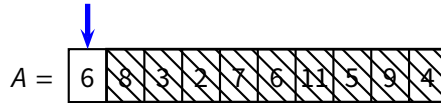
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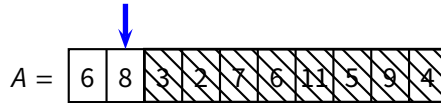
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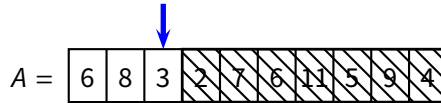
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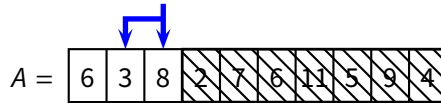
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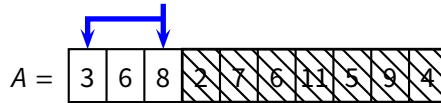
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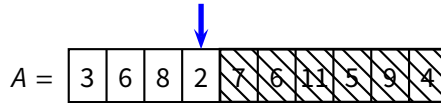
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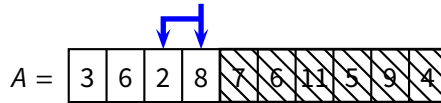
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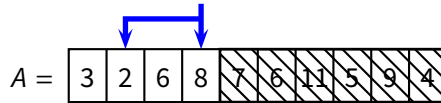
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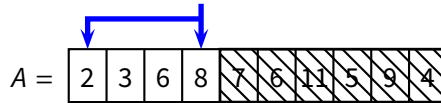
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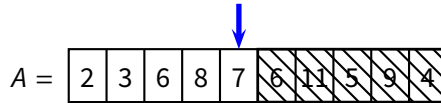
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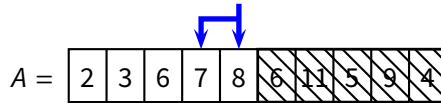
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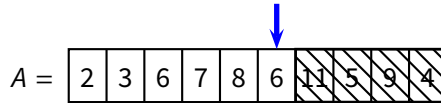
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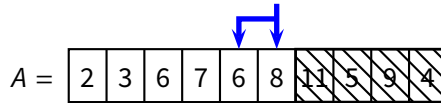
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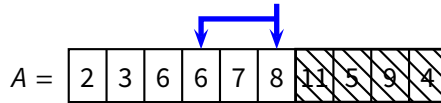
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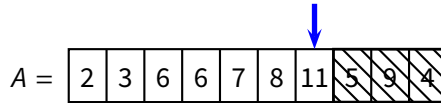
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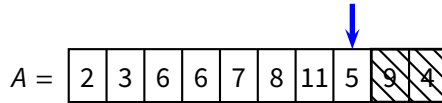
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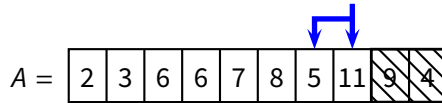
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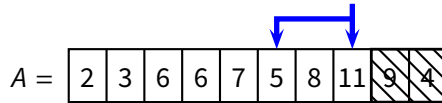
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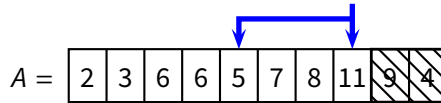
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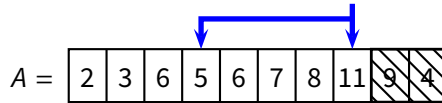
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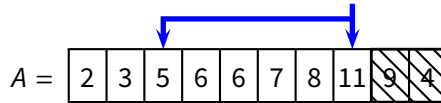
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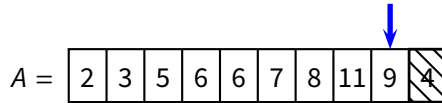
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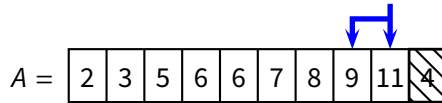
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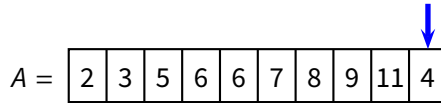
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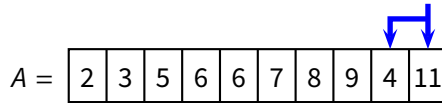
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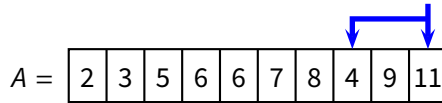
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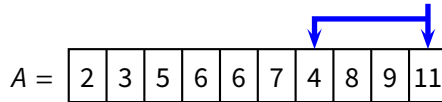
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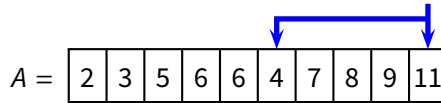
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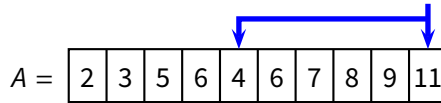
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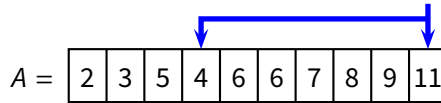
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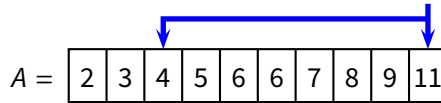
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INSERTION-SORT(A)

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1  for  $i = 2$  to  $\text{length}(A)$ 
2       $j = i$ 
3      while  $j > 1$  and  $A[j - 1] > A[j]$ 
4          swap  $A[j]$  and  $A[j - 1]$ 
5           $j = j - 1$ 
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- Is **INSERTION-SORT** *correct*?
- What is the time complexity of **INSERTION-SORT**?
- Can we do better?

Complexity of INSERTION-SORT

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- Outer loop (lines 1–5) runs exactly $n - 1$ times (with $n = length(A)$)
- What about the inner loop (lines 3–5)?
 - ▶ best, worst, and average case?

Complexity of INSERTION-SORT (2)

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- **Best case:**

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- **Best case:** the inner loop is *never* executed
 - ▶ what case is this?

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- **Best case:** the inner loop is *never* executed
 - ▶ what case is this?
- **Worst case:**

Complexity of INSERTION-SORT (2)

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1  for  $i = 2$  to  $length(A)$ 
2       $j = i$ 
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5           $j = j - 1$ 
```

- **Best case:** the inner loop is *never* executed
 - ▶ what case is this?
- **Worst case:** the inner loop is executed exactly $j - 1$ times for every iteration of the outer loop
 - ▶ what case is this?

Complexity of INSERTION-SORT (3)

- The worst-case complexity is when the inner loop is executed exactly $j - 1$ times, so

$$T(n) = \sum_{j=2}^n (j - 1)$$

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- If so, does it satisfy the conditions of the sorting problem?
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- We want ***a formal proof of correctness***
 - ▶ does not seem straightforward...

The Logic of Algorithmic Steps

Example 1: (straight-line program)

BIGGER(n)

1 // must return a value greater than n

2 $m = n * n + 1$

3 **return** m

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2  $m = n * n + 1$ 
3 return  $m$ 
```

Example 2: (branching)

SortTwo(A)

```
1 // must sort (in-place) an array of 2 elements
2 if  $A[1] > A[2]$ 
3      $t = A[1]$ 
4      $A[1] = A[2]$ 
5      $A[2] = t$ 
```

Example 3: (nested branching)

MAXTHREE(A)

```
1 // find the maximum value in an array of 3 elements
2 if A[1] > A[2]
3     if A[2] > A[3]
4         return A[1]
5     else return A[3]
6 else if A[3] > A[2]
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Is this algorithm correct?

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Is this algorithm correct?

Why?

Example 4: (second variant)

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Is this algorithm correct?

Prove it!

Example 5: (third variant)

```
MAXTHREE(a, b, c)
```

```
1 // find the maximum among 3 values
```

```
2 if a > b and a > c
```

```
3     return a
```

```
4 if b > c
```

```
5     return b
```

```
6 else return c
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Problem: what happens when we have *loops*?

Loop Invariants

- We formulate a *loop-invariant* condition C
 - ▶ C must remain true *through* the loop

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- Then, we only need to prove that the algorithm terminates

Loop Invariants (2)

- Formulation: this is where we try to be smart
 - ▶ *the invariant must reflect the structure of the algorithm*
 - ▶ it must be the basis to prove the correctness of the solution

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 - ▶ *the invariant must reflect the structure of the algorithm*
 - ▶ it must be the basis to prove the correctness of the solution
- Proof of validity (i.e., that C is indeed a loop invariant): typical *proof by induction*
 - ▶ **initialization:** we must prove that *the invariant C is true before entering the loop*
 - ▶ **maintenance:** we must prove that *if C is true at the beginning of a cycle **then** it remains true after one cycle*

Loop Invariant for INSERTION-SORT

INSERTION-SORT(A)

```
1  for  $i = 2$  to  $\text{length}(A)$ 
2       $j = i$ 
3      while  $j > 1$  and  $A[j - 1] > A[j]$ 
4          swap  $A[j]$  and  $A[j - 1]$ 
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- The main idea is to insert $A[i]$ in $A[1 \dots i - 1]$ so as to maintain a *sorted subsequence* $A[1 \dots i]$

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- The main idea is to insert $A[i]$ in $A[1..i-1]$ so as to maintain a *sorted subsequence* $A[1..i]$
- **Invariant:** (outer loop) *the subarray $A[1..i-1]$ consists of the elements originally in $A[1..i-1]$ in sorted order*

Loop Invariant for INSERTION-SORT (2)

INSERTION-SORT(A)

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```

- **Initialization:** $j = 2$, so $A[1..j - 1]$ is the single element $A[1]$
 - ▶ $A[1]$ contains the original element in $A[1]$
 - ▶ $A[1]$ is trivially sorted

Loop Invariant for INSERTION-SORT (3)

INSERTION-SORT(A)

```
1  for  $i = 2$  to  $length(A)$ 
2       $j = i$ 
3      while  $j > 1$  and  $A[j - 1] > A[j]$ 
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```

- **Maintenance:** informally, if $A[1..i-1]$ is a permutation of the original $A[1..i-1]$ and $A[1..i-1]$ is sorted (invariant), then *if* we enter the inner loop:
 - ▶ shifts the subarray $A[k..i-1]$ by one position to the right
 - ▶ inserts *key*, which was originally in $A[i]$ at its proper position $1 \leq k \leq i-1$, in sorted order

Loop Invariant for INSERTION-SORT (4)

INSERTION-SORT(A)

```
1  for  $i = 2$  to  $length(A)$ 
2       $j = i$ 
3      while  $j > 1$  and  $A[j - 1] > A[j]$ 
4          swap  $A[j]$  and  $A[j - 1]$ 
5           $j = j - 1$ 
```

Loop Invariant for INSERTION-SORT (4)

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- **Termination:** **INSERTION-SORT** terminates with $i = length(A) + 1$; the invariant states that
 - ▶ $A[1..i-1]$ is a permutation of the original $A[1..i-1]$
 - ▶ $A[1..i-1]$ is sorted

Given the termination condition, $A[1..i-1]$ is the whole A

So **INSERTION-SORT** is *correct*!

- You are given a problem P and an algorithm A
 - ▶ P formally defines a *correctness* condition
 - ▶ assume, for simplicity, that A consists of one loop
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(for all valid inputs)

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- ▶ prove that if C holds right before the first instruction of the loop, then it holds also at the end of the loop

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5. Prove that $X \wedge C \Rightarrow P$, which means that A is correct

Exercise: Analyze Selection-Sort

SELECTION-SORT(A)

```
1   $n = \text{length}(A)$ 
2  for  $i = 1$  to  $n - 1$ 
3       $\text{smallest} = i$ 
4      for  $j = i + 1$  to  $n$ 
5          if  $A[j] < A[\text{smallest}]$ 
6               $\text{smallest} = j$ 
7      swap  $A[i]$  and  $A[\text{smallest}]$ 
```

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■ Correctness?

- ▶ loop invariant?

■ Complexity?

- ▶ worst, best, and average case?

Exercise: Analyze Bubblesort

BUBBLESORT(A)

```
1  for  $i = 1$  to  $\text{length}(A)$   
2      for  $j = \text{length}(A)$  downto  $i + 1$   
3          if  $A[j] < A[j - 1]$   
4              swap  $A[j]$  and  $A[j - 1]$ 
```

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BUBBLESORT(A)

```
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