Basics of Complexity Analysis: The RAM Model and the Growth of Functions

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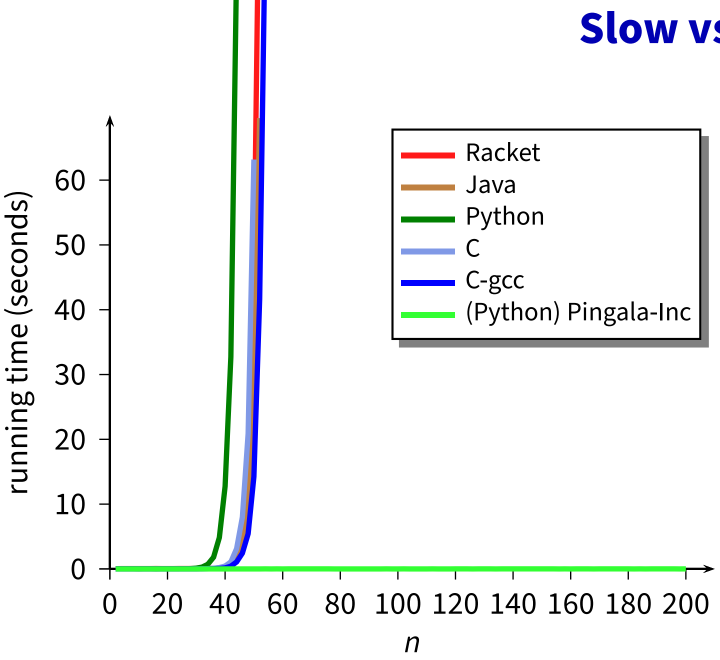
Outline

- Informal analysis of two Pingala algorithms
- The *random-access machine* model
- Measure of complexity
- Characterizing functions with their asymptotic behavior
- Big-O, omega, and theta notations

Slow vs. Fast Pingala



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  - ***in general***
  - in a way that is ***specific to the algorithms***
  - but ***independent of implementation details***

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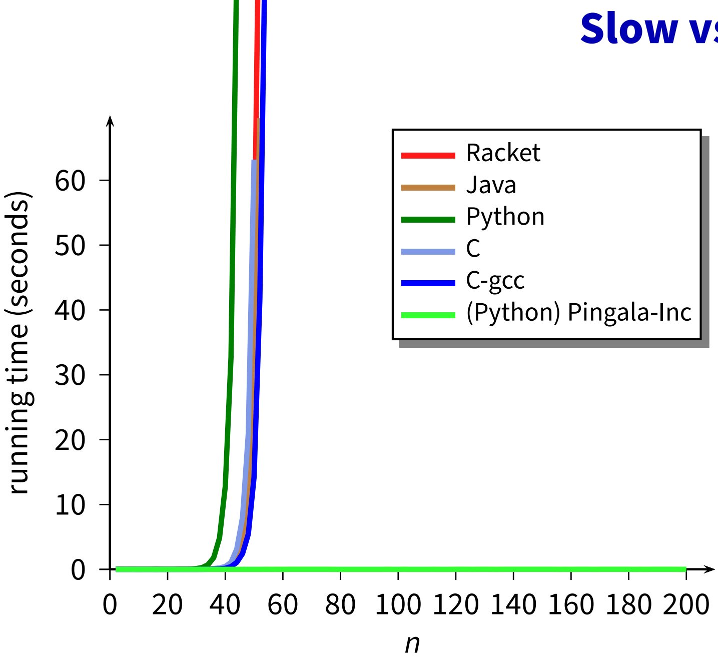

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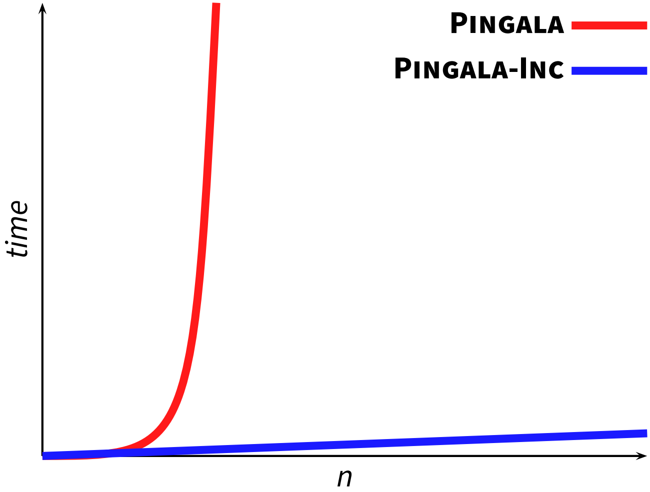

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  - subroutine call
- A ***basic step*** in the RAM model takes a ***constant time***

Analysis in the RAM Model

```
PINGALA-INC(n)
if n ≤ 2
    return n
pprev = 1
prev = 2
for i = 3 to n
    P = prev + pprev
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cost	*times* $(n > 2)$
c_1	1
c_2	0
c_3	1
c_4	1
c_5	$n-1$
c_6	$n-2$
c_7	$n-2$
c_8	$n-2$
c_9	1

$$T(n) = c_1 + c_3 + c_4 + c_9 + (n-1)c_5 + (n-2)(c_6 + c_7 + c_8)$$

Constant Factors

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- ***We do not care about the specific costs of each basic step***
  - these costs are likely to vary significantly with languages, implementations, and processors
  - we simplify our model by effectively considering only the maximal cost of any basic step
  - so, we assume $c_1 = c_2 = c_3 = \cdots = c_i$

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$T(n) = nC_1 + C_2$ $\Rightarrow$ $T(n)$ is a *linear function* of n

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- A ***basic step*** in the RAM model takes a ***constant time***
  - "constant" means ***independent of the input size***
- The *specific* constant is a ***technological factor***
- ***Technology changes***

…so ***we ignore any specific multiplicative or additive constants***

…effectively we allow for ***any scaling factor***

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```
FIND(A, x)
for i = 1 to length(A)
    if A[i] == x
        return TRUE
return FALSE
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$$T(n) = Cn$$

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FINDEQUALS(A)
for i = 1 to length(A) − 1
    for j = i + 1 to length(A)
        if A[i] == A[j]
            return TRUE
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FindEquals(A)
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return FALSE
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$$T(n) = C\frac{n(n-1)}{2}$$

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- We care only about the ***asymptotic order of growth*** of $T(n)$
  - so we ignore lower-order terms

    Example:

    Algorithm 1 costs $T_1(n) = 100n + 3000$ basic steps

    Algorithm 2 costs $T_2(n) = 0.02n^2 + 2$ basic steps

    Which is best?

O-Notation

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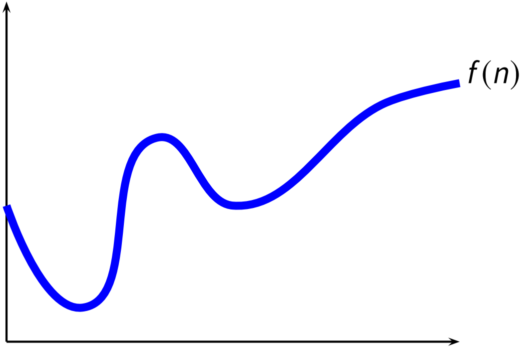

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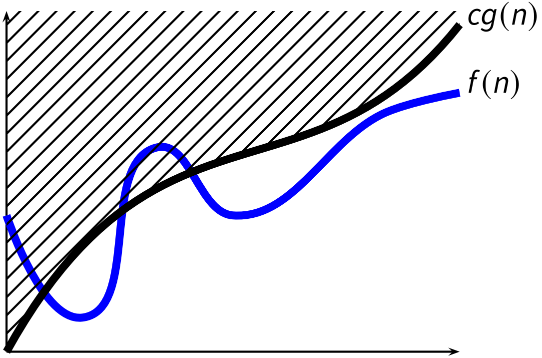

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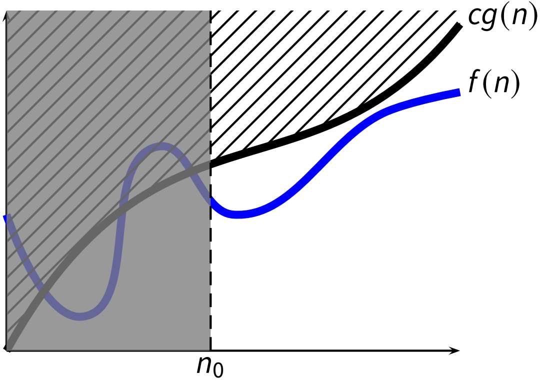

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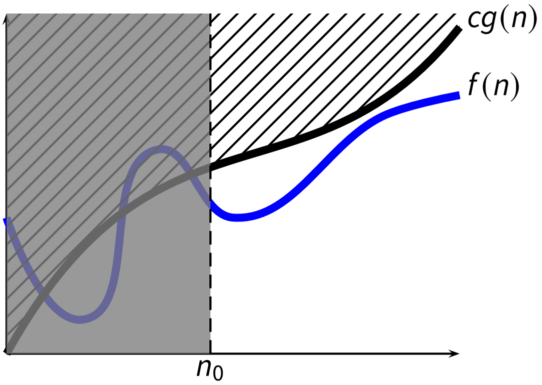

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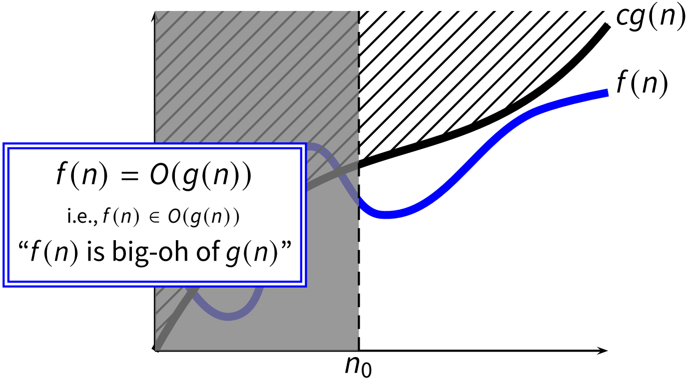

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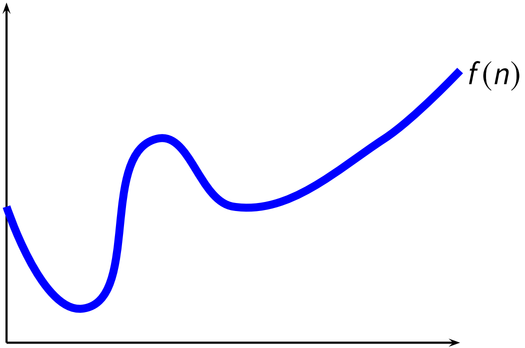

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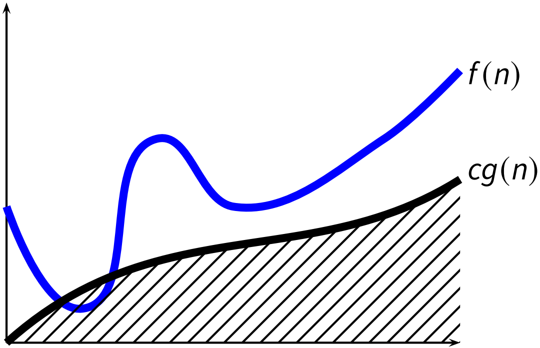

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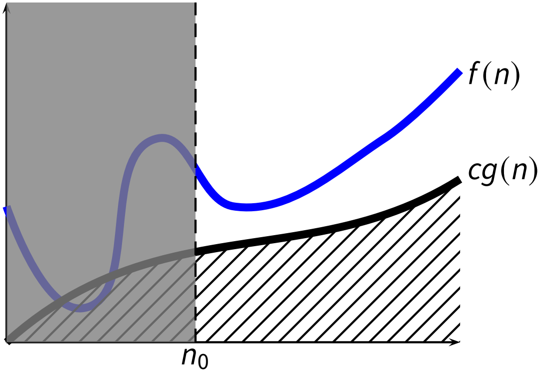

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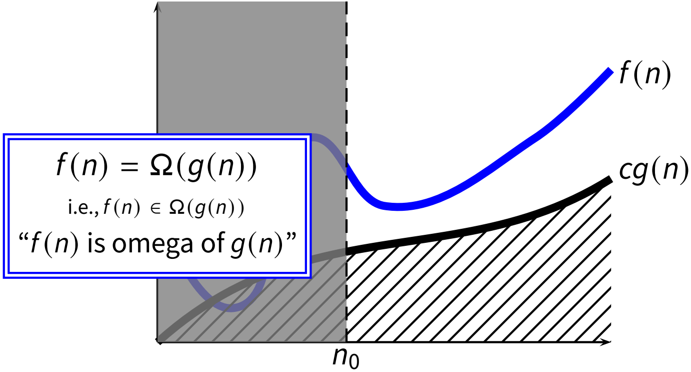

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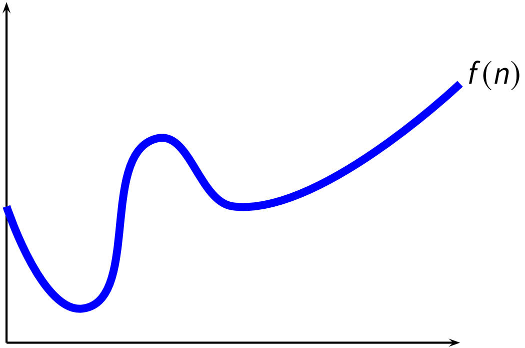

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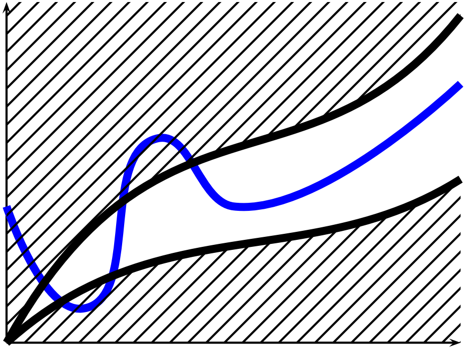

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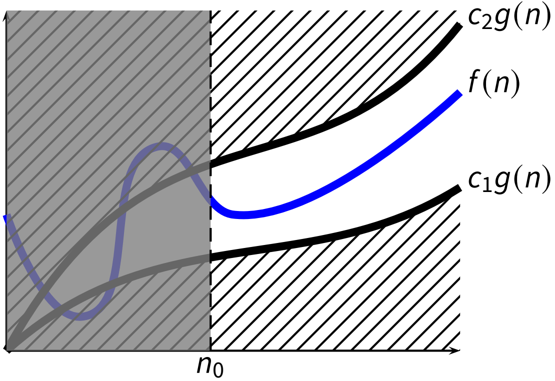

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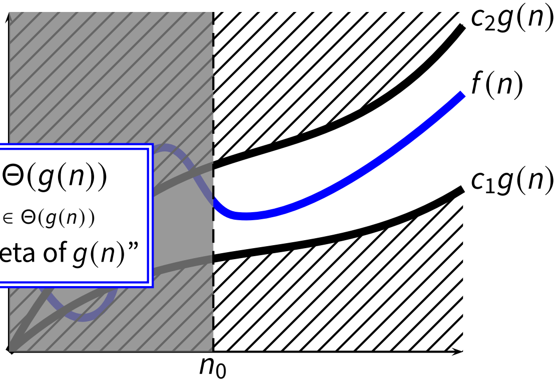

$f(n) = \Theta(g(n))$

i.e., $f(n) \in \Theta(g(n))$

"$f(n)$ is theta of $g(n)$"

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- We use the o-notation to denote upper bounds that are *not* asymtotically tight. So, given a function $g(n)$, we define the family of functions $o(g(n))$

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  In fact, the fundamental *prime number theorem* says that

$$\lim_{n\to\infty} \frac{\pi(n) \ln n}{n} = 1$$

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- $T(n) = n\log n + n\sqrt{n} \quad \Rightarrow T(n) = \Theta(n\sqrt{n})$
- $T(n) = 2^{\frac{n}{6}} + n^7$

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- $T(n) =$ complexity of **Pingala-Inc**

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- $T(n) =$ complexity of **Pingala-Inc** $\quad \Rightarrow T(n) = \Theta(n)$
- We characterize the behavior of $T(n)$ *in the limit*
- The Θ-notation is an ***asymptotic notation***

Examples

- $f(n) = n^2 + 10n + 100$

Examples

- $f(n) = n^2 + 10n + 100 \quad \Longrightarrow f(n) = O(n^2)$

Examples

- $f(n) = n^2 + 10n + 100 \quad \Rightarrow f(n) = O(n^2) \quad \Rightarrow f(n) = O(n^3)$

Examples

- $f(n) = n^2 + 10n + 100 \quad \Rightarrow f(n) = O(n^2) \quad \Rightarrow f(n) = O(n^3)$
- $f(n) = n + 10 \log n$

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- $f(n) = n + 10 \log n \quad \Rightarrow f(n) = O(2^n)$

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- $f(n) = n \log n + n\sqrt{n}$

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Examples

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Examples

- $n^2 - 10n + 100 = O(n \log n)$?

Examples

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Example

- So, what is the complexity of **FindEquals**?

```
FindEquals(A)
for i = 1 to length(A) − 1
    for j = i + 1 to length(A)
        if A[i] == A[j]
            return TRUE
return FALSE
```

Example

- So, what is the complexity of **FINDEQUALS**?

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FINDEQUALS(A)
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return FALSE
```

$$T(n) = \Theta(n^2)$$

- $n = \mathit{length}(A)$ is the ***size of the input***
- we measure the ***worst-case complexity***

Θ, O, and Ω as Relations

- *Theorem:* for any two functions $f(n)$ and $g(n)$,
  $f(n) = \Omega(g(n)) \wedge f(n) = O(g(n)) \Leftrightarrow f(n) = \Theta(g(n))$

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- When $f(n) = \Omega(g(n))$ we say that $g(n)$ is a ***lower bound*** for $f(n)$

Θ, O, and Ω as Anonymous Functions

- We can use the Θ-, O-, and Ω-notation to represent anonymous (unknown or unsecified) functions
  E.g.,

  $$f(n) = 10n^2 + O(n)$$

  means that $f(n)$ is equal to $10n^2$ plus a function we don't know or we don't care to know that is asymptotically at most linear in n.

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  $n^2 + O(n) - 1 = O(n^2)$? YES

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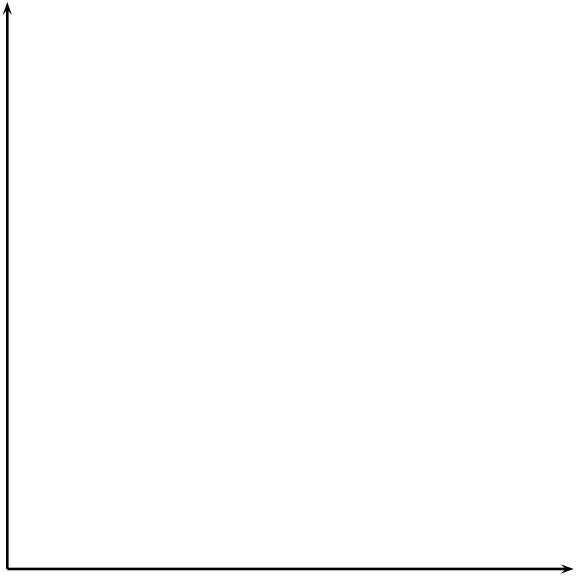

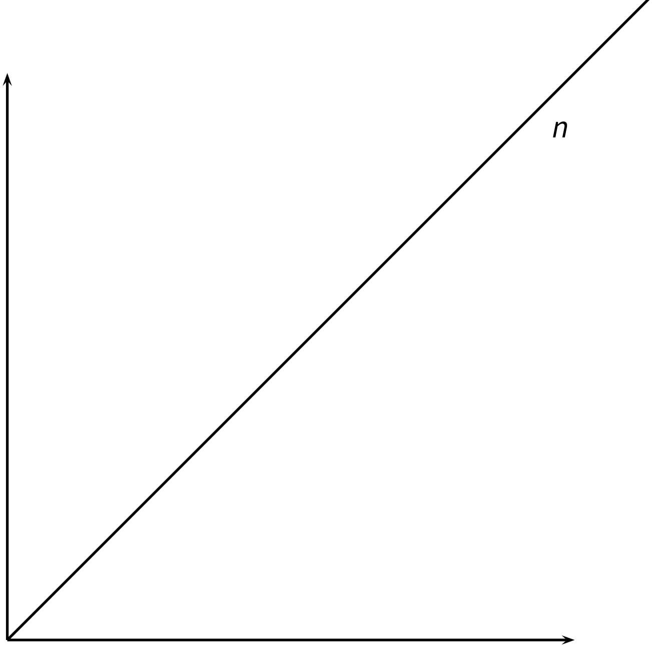
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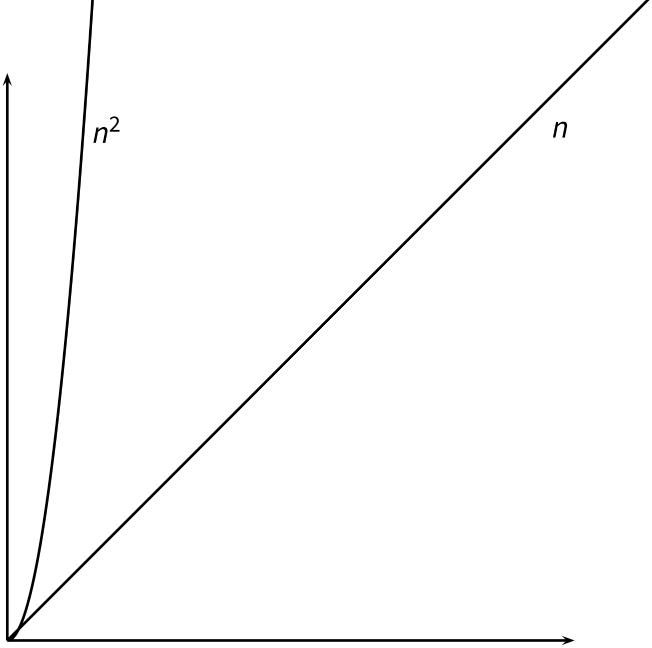
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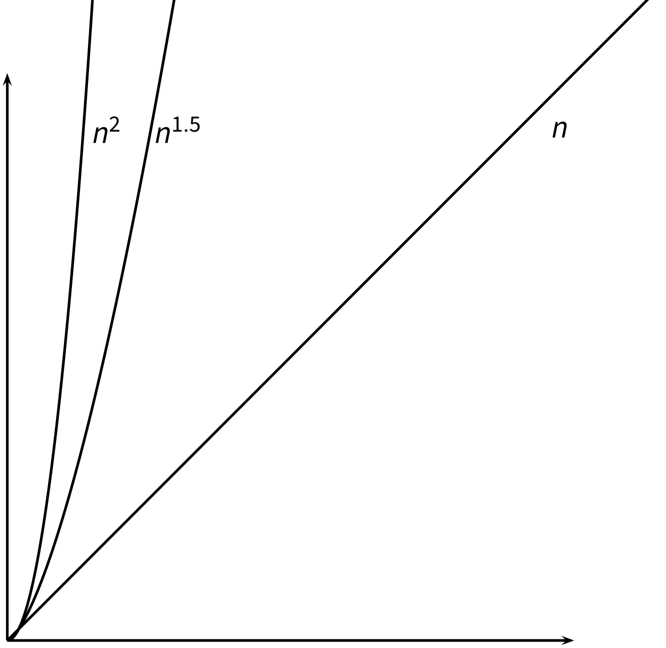
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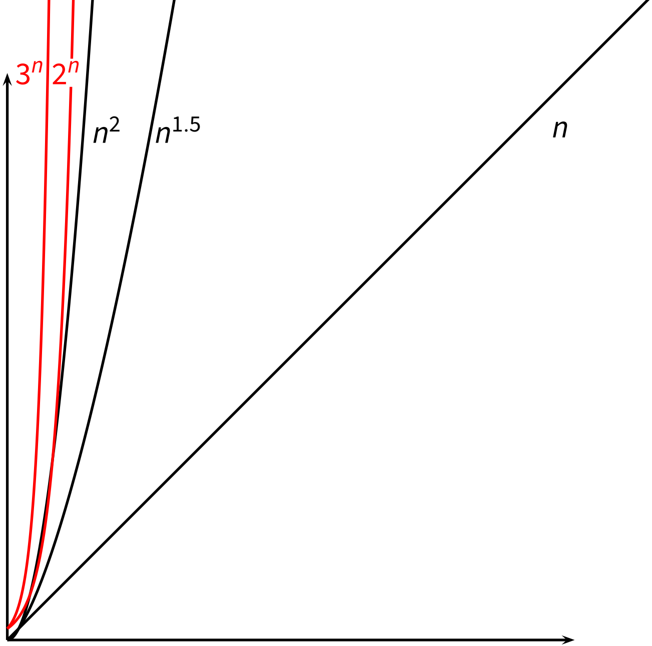
$n^{1.5}$
n



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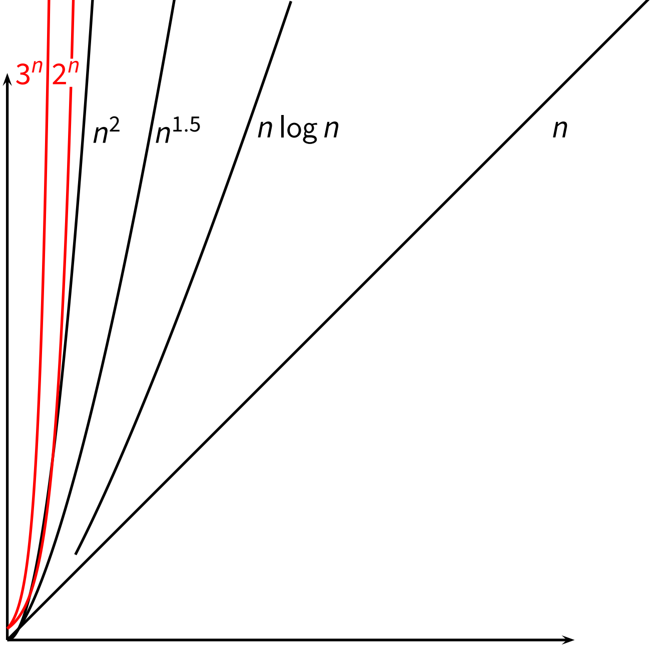
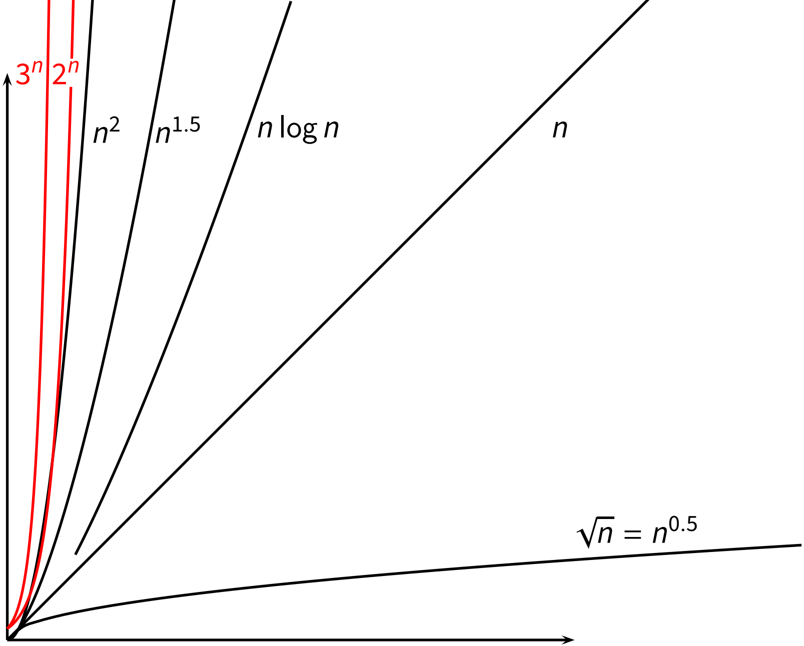
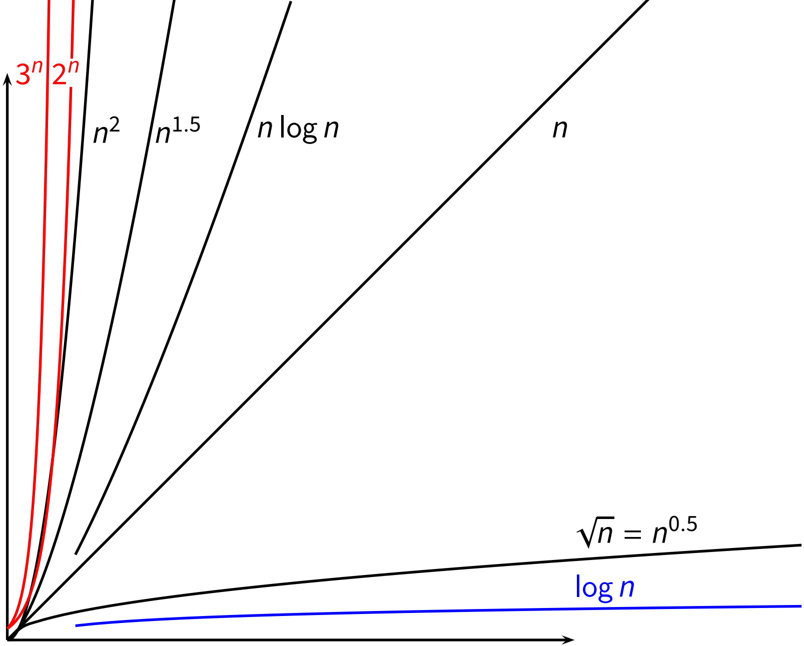
2^n
n^2
$n^{1.5}$
n

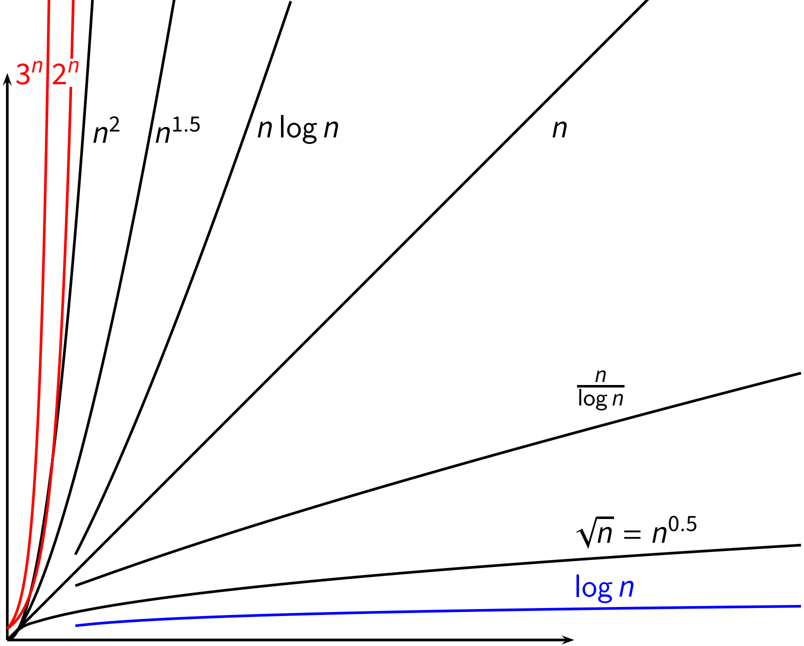

3^n
2^n
n^2
$n^{1.5}$
$n \log n$
n



3^n
2^n
n^2
$n^{1.5}$
$n \log n$
n
$\sqrt{n} = n^{0.5}$



3^n
2^n
n^2
$n^{1.5}$
$n \log n$
n
$\sqrt{n} = n^{0.5}$
$\log n$



3^n
2^n
n^2
$n^{1.5}$
$n \log n$
n
$\frac{n}{\log n}$
$\sqrt{n} = n^{0.5}$
$\log n$