Elementary Data Structures and Hash Tables

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Outline

- Common concepts and notation
- Stacks
- Queues
- Linked lists
- Trees
- Direct-access tables
- Hash tables

Concepts

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- A data structure stores ***data*** and possibly ***meta-data***
  - e.g., a *heap* needs an array A to store the keys, plus a variable $A.\mathit{heap\text{-}size}$ to remember how many elements are in the heap

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  - **PUSH**(S, x) pushes the value x onto the stack S
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- *Implementation*
  - using an array
  - using a linked list
  - …

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    return TRUE
else return FALSE
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else return FALSE
```

```
PUSH(S,x)
$S.top = S.top + 1$
$S[S.top] = x$
```

```
POP(S)
if STACK-EMPTY($S$)
        error "underflow"
else $S.top = S.top - 1$
        return $S[S.top + 1]$
```

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  - **ENQUEUE**(Q, x) adds element x at the back of queue Q
  - **DEQUEUE**(Q) extracts the element at the head of queue Q
- *Implementation*
  - Q is an array of fixed length $Q.length$
    - i.e., Q holds at most $Q.length$ elements
    - enqueueing more than Q elements causes an “overflow” error
  - $Q.head$ is the position of the “head” of the queue
  - $Q.tail$ is the first empty position at the tail of the queue

Enqueue

```
ENQUEUE(Q,x)
if $Q.queue\text{-}full$
    error "overflow"
else $Q[Q.tail] = x$
    if $Q.tail < Q.length$
        $Q.tail = Q.tail + 1$
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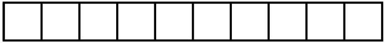

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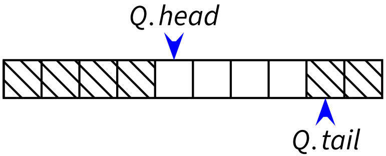

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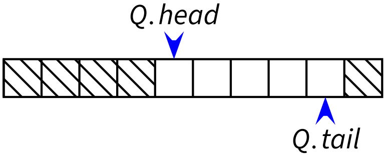

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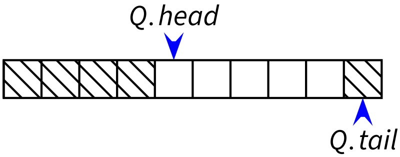

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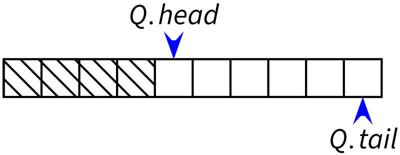

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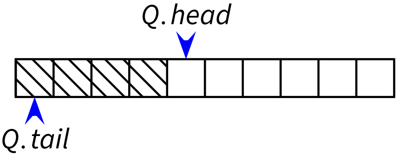

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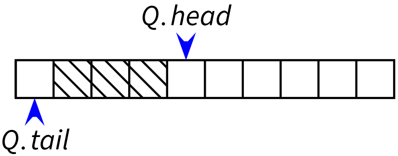

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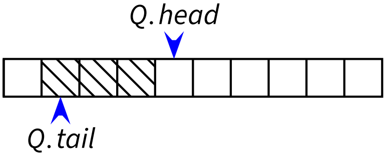

Dequeue

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DEQUEUE(Q)
if $Q.queue\text{-}empty$
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else $x = Q[Q.head]$
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        return $x$
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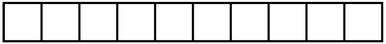

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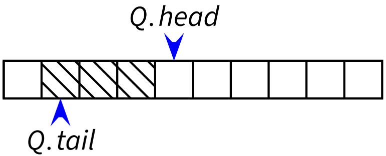

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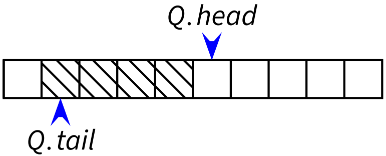

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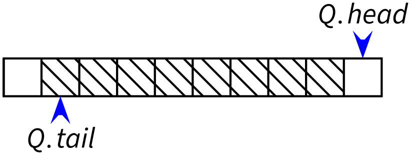

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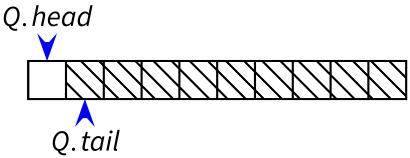

Linked List

- *Interface*
  - **LIST-INSERT**(L, x) adds element x at beginning of a list L
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  - **LIST-SEARCH**(L, k) finds an element whose key is k in a list L

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- *Implementation*
  - a *doubly-linked* list
  - each element x has two “links” $x.prev$ and $x.next$ to the previous and next elements, respectively
  - each element x holds a key $x.key$
  - it is convenient to have a dummy “sentinel” element $L.nil$

Linked List With a "Sentinel"

```
List-Init($L$)
    $L.nil.prev = L.nil$
    $L.nil.next = L.nil$
```

```
List-Insert($L, x$)
    $x.next = L.nil.next$
    $L.nil.next.prev = x$
    $L.nil.next = x$
    $x.prev = L.nil$
```

```
List-Search($L, k$)
    $x = L.nil.next$
    while $x \neq L.nil \land x.key \neq k$
        $x = x.next$
    return $x$
```

Complexity

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Algorithm	*Complexity*

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Stack-Empty	

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LIST-INSERT	

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  - ***hash tables***

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```
DIRECT-ADDRESS-INSERT($T, k$)
$T[k] = $ TRUE
```

```
DIRECT-ADDRESS-DELETE($T, k$)
$T[k] = $ FALSE
```

```
DIRECT-ADDRESS-SEARCH($T, k$)
return $T[k]$
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- *Can we have the benefits of a direct-address table but with a table of reasonable size?*

Hash Table

- *Idea*
  - use a table T with $|T| \ll |U|$
  - map each key $k \in U$ to a position in T, using a ***hash function***

$$h : U \longrightarrow \{1, \ldots, |T|\}$$

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HASH-INSERT$(T, k)$
    $T[h(k)] =$ TRUE
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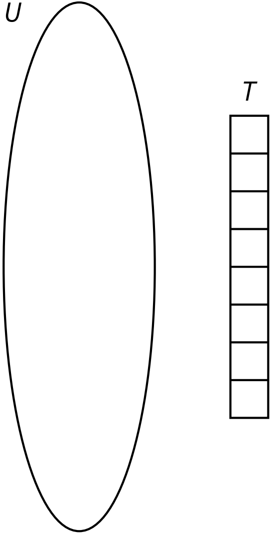
What if two distinct keys $k_1 \neq k_2$ *collide?* (I.e., $h(k_1) = h(k_2)$)

Hash Table



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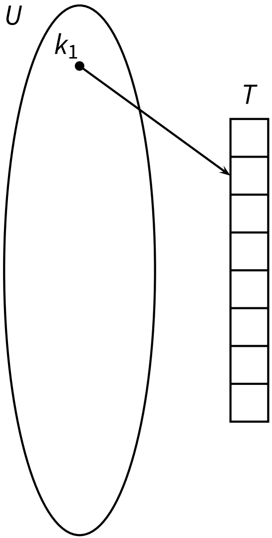

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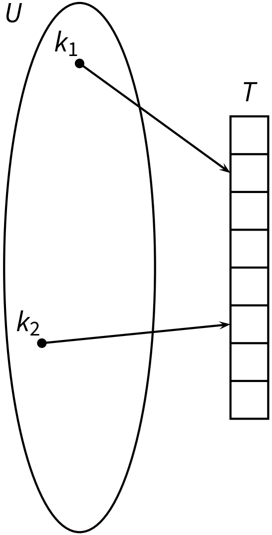

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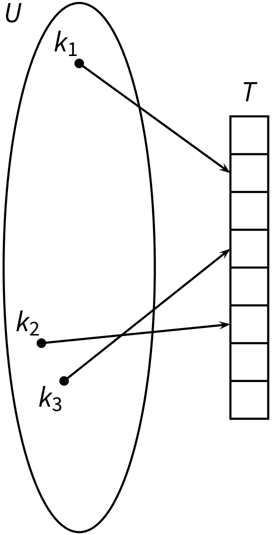

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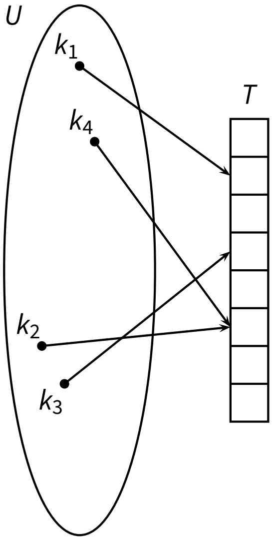

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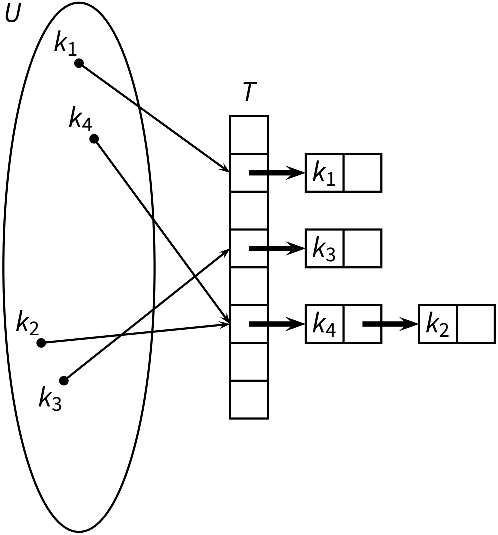

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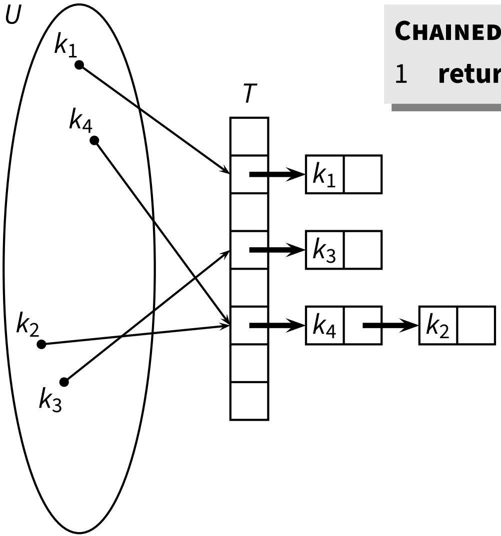

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CHAINED-HASH-INSERT$(T, k)$
    return LIST-INSERT$(T[h(k)], k)$
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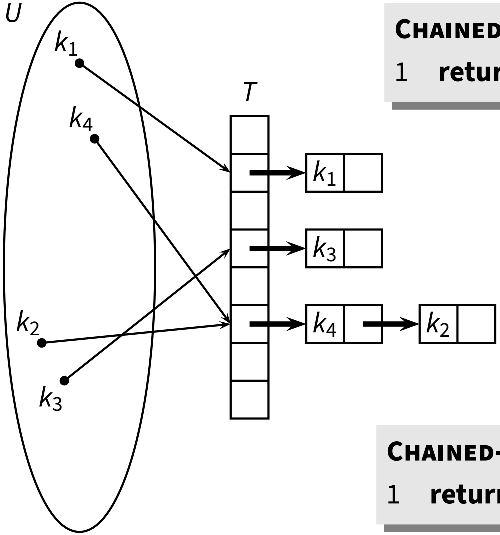

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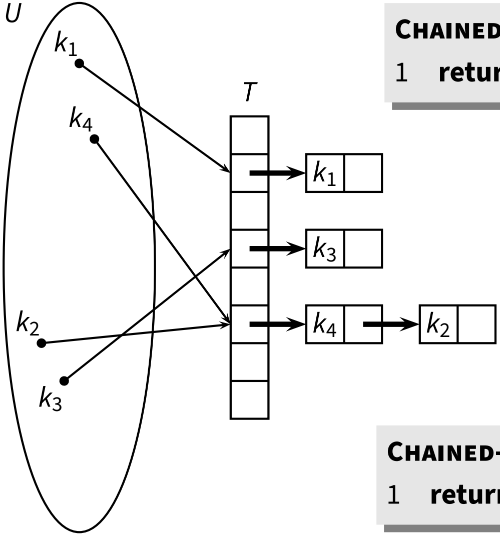

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load factor

$$\alpha = \frac{n}{|T|}$$

```
CHAINED-HASH-SEARCH$(T, k)$
    return LIST-SEARCH$(T[h(k)], k)$
```

Analysis

- We assume ***uniform hashing*** for our hash function $h : U \longrightarrow \{1 \ldots |T|\}$ (where $|T| = T.length$)

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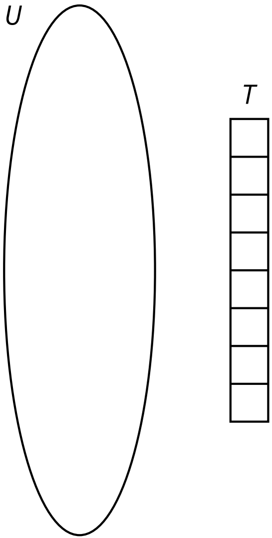
- Therefore, the complexity of **CHAINED-HASH-SEARCH** is

  $$\boxed{\Theta(1 + \alpha)}$$

Open-Address Hash Table



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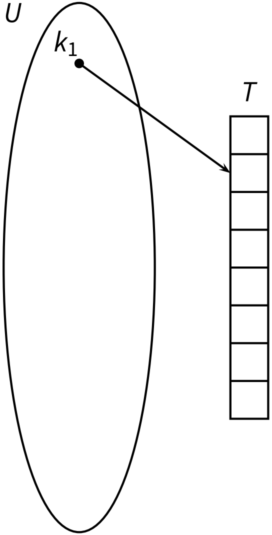

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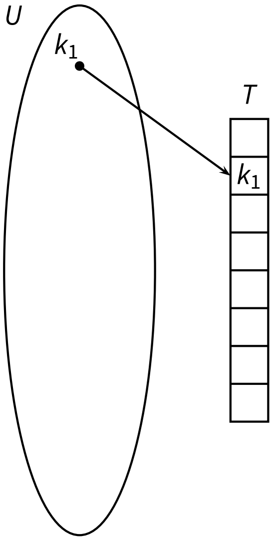

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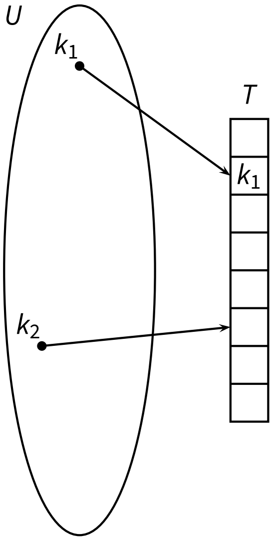

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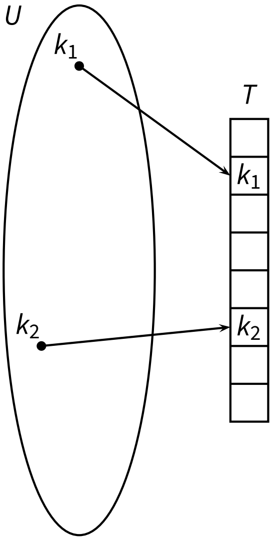

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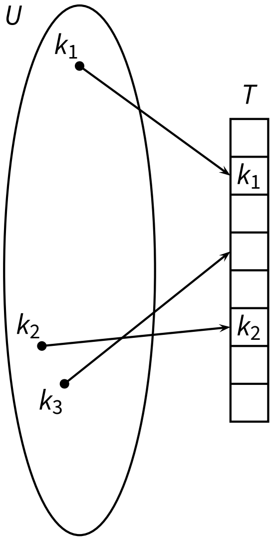

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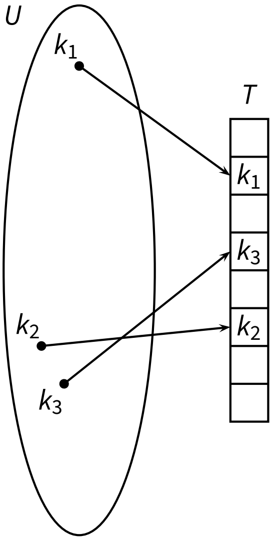

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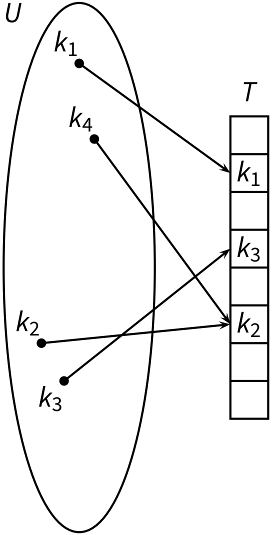

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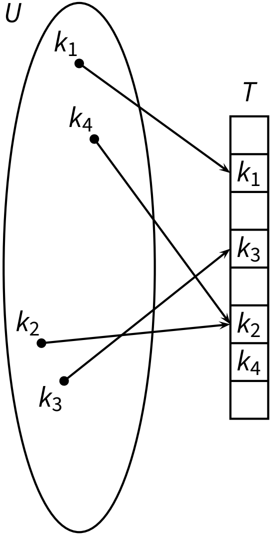

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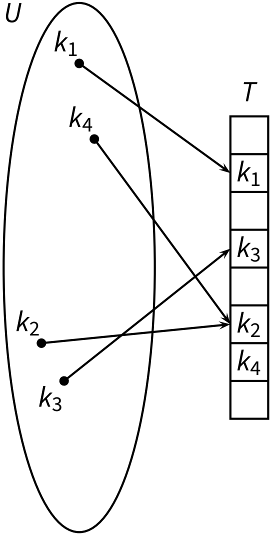

```
HASH-INSERT(T, k)
j = h(k)
for i = 1 to T.length
        if T[j] == NIL
                T[j] = k
                return j
        elseif j < T.length
                j = j + 1
        else j = 1
error "overflow"
```

Open-Addressing (2)

- *Idea:* instead of using linked lists, we can store all the elements in the table
  - this implies $\alpha \leq 1$

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- A sequential “probe” may not be optimal
    - can you figure out why?

Open-Addressing (3)

```
HASH-INSERT$(T, k)$
for $i = 1$ to $T.length$
$j = h(k, i)$
    if $T[j]$ == NIL
        $T[j] = k$
        return $j$
error "overflow"
```

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HASH-INSERT($T, k$)
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error “overflow”
```

- Notice that $h(k, \cdot)$ must be a ***permutation***
  - i.e., $h(k, 1), h(k, 2), \ldots, h(k, |T|)$ must cover the entire table T