

Divide-and-Conquer Algorithms

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- Merging (or set union)
- Searching
- Sorting
- Multiplying
- Computing the *median*

Merging (Set Union)

■ **Input:** sequences $A = \langle a_1, a_2, \dots, a_n \rangle$ and $B = \langle b_1, b_2, \dots, b_m \rangle$

Output: a sequence $X = \langle x_1, x_2, \dots, x_\ell \rangle$ such that

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■ **Example:**

$A = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10 \rangle$

$B = \langle 51, 21, 14, 15, 27, 31, 2 \rangle$

$X =$

■ **Input:** sequences $A = \langle a_1, a_2, \dots, a_n \rangle$ and $B = \langle b_1, b_2, \dots, b_m \rangle$

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■ **Example:**

$A = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10 \rangle$

$B = \langle 51, 21, 14, 15, 27, 31, 2 \rangle$

$X = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10, 15, 27, 2 \rangle$

A Simple Merge Algorithm

- Algorithm strategy

■ Algorithm strategy

- ▶ iterate through every position i , first through A , and then B
- ▶ output a_i if a_i is not in $\langle a_1, a_2, \dots, a_{i-1} \rangle$
- ▶ output b_i if b_i is not in $\langle a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_{i-1} \rangle$

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MERGESIMPLE(A, B)

```
1  for  $i = 1$  to  $\text{length}(A)$ 
2      if not FIND( $A[1..i-1], A[i]$ )
3          output  $A[i]$ 
4  for  $i = 1$  to  $\text{length}(B)$ 
5      if not FIND( $A, B[i]$ ) and not FIND( $B[1..i-1], B[i]$ )
6          output  $B[i]$ 
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MERGESIMPLE(A, B)

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2 **if not** **FIND**($A[1..i-1], A[i]$)

3 output $A[i]$

4 **for** $i = 1$ **to** $\text{length}(B)$

5 **if not** **FIND**($A, B[i]$) **and not** **FIND**($B[1..i-1], B[i]$)

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```

let $n = length(A) + length(B)$

$$T(n) = \sum_{i=1}^{length(A)} T_{\mathbf{FIND}}(i) + \sum_{i=1}^{length(B)} (T_{\mathbf{FIND}}(i) + T_{\mathbf{FIND}}(length(A)))$$

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$$T(n) = \sum_{i=1}^n T_{\mathbf{FIND}}(i)$$

■ **Input:** a sequence A and a value key

Output: TRUE if A contains key , or FALSE otherwise

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FIND(A, key)

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2      if  $A[i] == key$ 
3          return TRUE
4  return FALSE
```

FIND($A, begin, end, key$)

```
1  for  $i = begin$  to  $end$ 
2      if  $A[i] == key$ 
3          return TRUE
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```
FINDINLIST( $A, key$ )
```

```
1  $item = first(A)$ 
```

```
2 while  $item \neq last(A)$ 
```

```
3     if  $value(item) == key$ 
```

```
4         return TRUE
```

```
5      $item = next(item)$ 
```

```
6 return FALSE
```

■ **Input:** a sequence A and a value key

Output: TRUE if A contains key , or FALSE otherwise

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Complexity of MERGESIMPLE

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4 **for** $i = 1$ **to** $\text{length}(B)$

5 **if not** **FIND**($A, B[i]$) **and not** **FIND**($B[1..i-1], B[i]$)

6 output $B[i]$

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$$T(n) = \sum_{i=1}^n T_{\mathbf{FIND}}(i)$$

$$T(n) = \sum_{i=1}^n O(i) = O\left(\frac{n(n+1)}{2}\right) = O(n^2)$$

- **Input:** a *sorted* sequence A and a value key
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BINARYSEARCH(A, key)

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1  first = 1
2  last = length( $A$ )
3  while first ≤ last
4      middle =  $\lceil (first + last) / 2 \rceil$ 
5      if  $A[middle] == key$ 
6          return TRUE
7      elseif first = last
8          return FALSE
9      elseif  $A[middle] > key$ 
10         last = middle - 1
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BINARYSEARCH(*A*, *key*)

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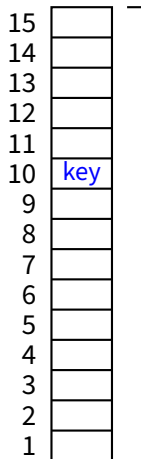
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15	
14	
13	
12	
11	
10	key
9	
8	
7	
6	
5	
4	
3	
2	
1	

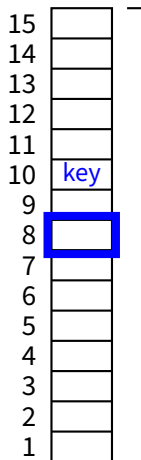
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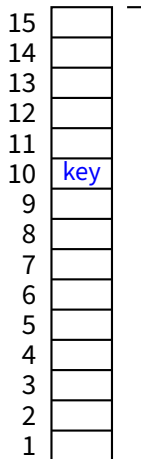
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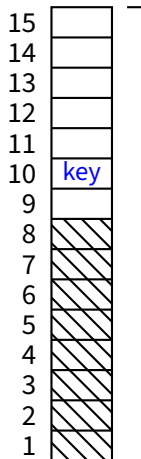
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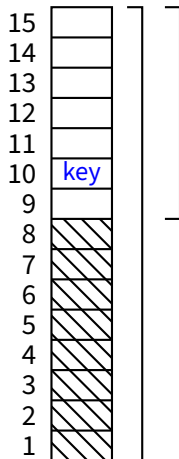
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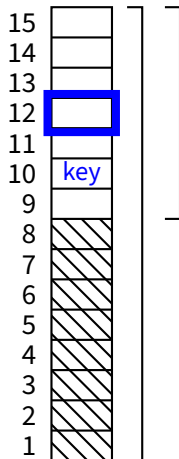
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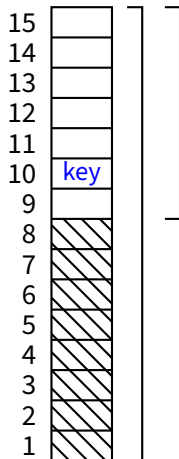
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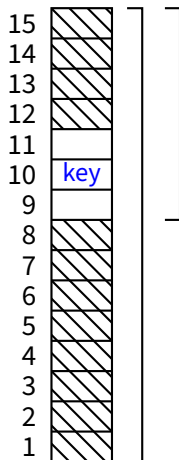
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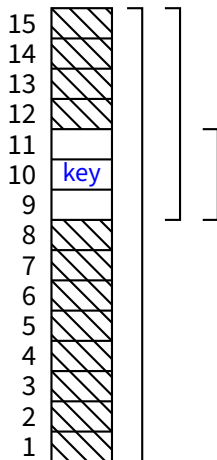
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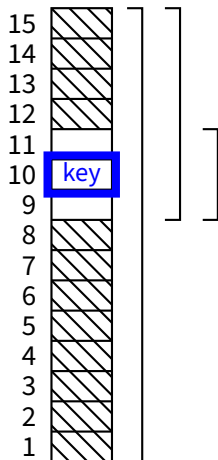
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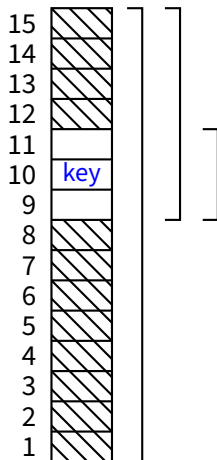
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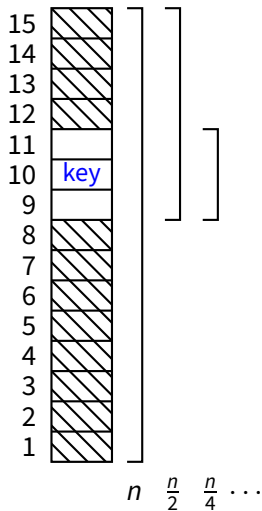


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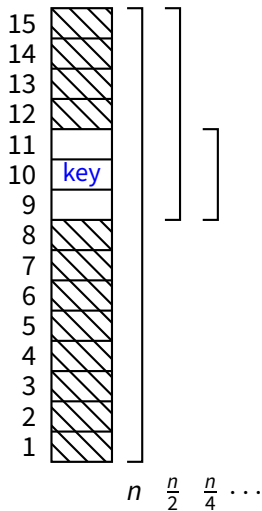
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```

$$T(n) = O(\log n)$$



- A slightly different problem:

Input: two *sorted* sequences $A = \langle a_1, a_2, \dots, a_n \rangle$ and $B = \langle b_1, b_2, \dots, b_m \rangle$, where $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_m$

Output: a sequence $X = \langle x_1, x_2, \dots, x_\ell \rangle$ such that

- ▶ every element of A appears once in X
- ▶ every element of B appears once in X
- ▶ every element of X appears in A or in B or in both

MERGESIMPLE2(A, B)

```
1  for  $i = 1$  to  $length(A)$ 
2      if not BINARYSEARCH( $A[1..i-1], A[i]$ )
3          output  $A[i]$ 
4  for  $i = 1$  to  $length(B)$ 
5      if not BINARYSEARCH( $A, B[i]$ )
6          and not BINARYSEARCH( $B[1..i-1], B[i]$ )
7          output  $B[i]$ 
```

A Better Merge Algorithm

MERGESIMPLE2(A, B)

```
1  for  $i = 1$  to  $length(A)$ 
2      if not BINARYSEARCH( $A[1..i-1], A[i]$ )
3          output  $A[i]$ 
4  for  $i = 1$  to  $length(B)$ 
5      if not BINARYSEARCH( $A, B[i]$ )
6          and not BINARYSEARCH( $B[1..i-1], B[i]$ )
7          output  $B[i]$ 
```

$$T(n) = \sum_{i=1}^n O(\log i) =$$

MERGESIMPLE2(A, B)

```
1  for  $i = 1$  to  $\text{length}(A)$ 
2      if not BINARYSEARCH( $A[1..i-1], A[i]$ )
3          output  $A[i]$ 
4  for  $i = 1$  to  $\text{length}(B)$ 
5      if not BINARYSEARCH( $A, B[i]$ )
6          and not BINARYSEARCH( $B[1..i-1], B[i]$ )
7          output  $B[i]$ 
```

$$T(n) = \sum_{i=1}^n O(\log i) = O(n \log n)$$

A Better Merge Algorithm

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$$T(n) = \sum_{i=1}^n O(\log i) = O(n \log n)$$

Better than $O(n^2)$, but can we do even better than $O(n \log n)$?

An Even Better Merge Algorithm

- *Intuition: A and B are sorted*

e.g.

$A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$

$B = \langle 1, 5, 6, 7, 34, 35, 40, 41, 43 \rangle$

An Even Better Merge Algorithm

- *Intuition: A and B are sorted*

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so just like in **BINARYSEARCH** I can avoid looking for an element x if the *first* element I see is $y > x$

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$B = \langle 1, 5, 6, 7, 34, 35, 40, 41, 43 \rangle$

so just like in **BINARYSEARCH** I can avoid looking for an element x if the *first* element I see is $y > x$

- High-level algorithm strategy

- ▶ step through every position i of A and every position j of B
- ▶ output a_i and advance i if $a_i \leq b_j$ or if j is beyond the end of B
- ▶ output b_j and advance j if $a_i \geq b_j$ or if i is beyond the end of A

MERGE Algorithm

A

3	7	12	13	34	37	70	75	80
---	---	----	----	----	----	----	----	----

B

1	5	6	7	34	35	40	41	43
---	---	---	---	----	----	----	----	----

MERGE Algorithm

$i = 1$



A	3	7	12	13	34	37	70	75	80
---	---	---	----	----	----	----	----	----	----

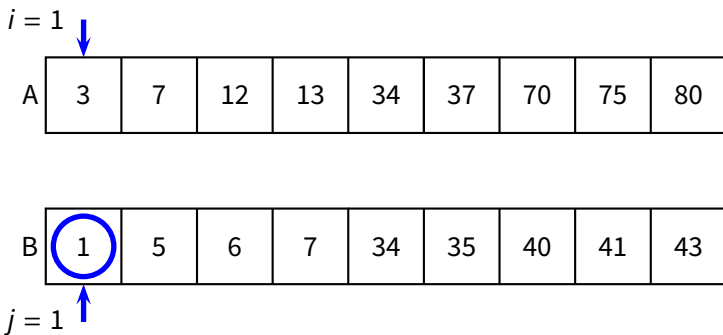
B	1	5	6	7	34	35	40	41	43
---	---	---	---	---	----	----	----	----	----

$j = 1$



Output:

MERGE Algorithm



Output:

MERGE Algorithm

$i = 1$



A	3	7	12	13	34	37	70	75	80
---	---	---	----	----	----	----	----	----	----

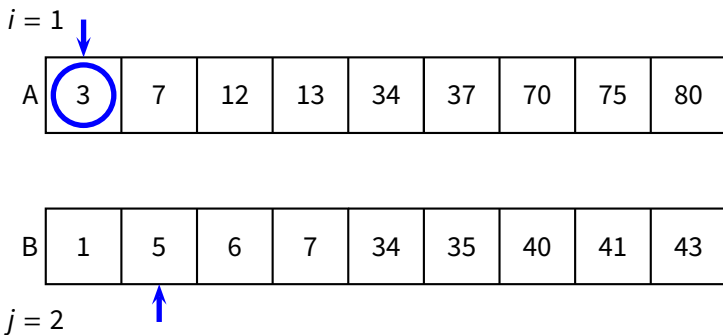
B	1	5	6	7	34	35	40	41	43
---	---	---	---	---	----	----	----	----	----

$j = 2$



Output: 1


MERGE Algorithm



Output: 1

MERGE Algorithm

$i = 2$

									
A	3	7	12	13	34	37	70	75	80

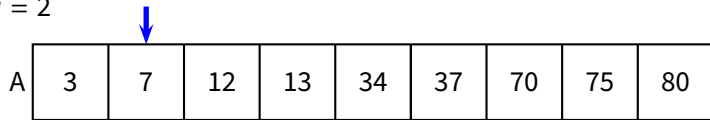
B	1	5	6	7	34	35	40	41	43
---	---	---	---	---	----	----	----	----	----

$j = 2$

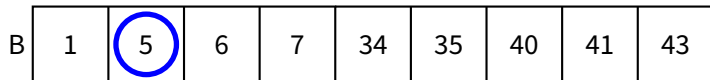
Output: 13

MERGE Algorithm

$i = 2$



A	3	7	12	13	34	37	70	75	80
---	---	---	----	----	----	----	----	----	----




B	1	5	6	7	34	35	40	41	43
---	---	---	---	---	----	----	----	----	----

$j = 2$

Output: 13

MERGE Algorithm

$i = 2$



A	3	7	12	13	34	37	70	75	80
---	---	---	----	----	----	----	----	----	----

B	1	5	6	7	34	35	40	41	43
---	---	---	---	---	----	----	----	----	----

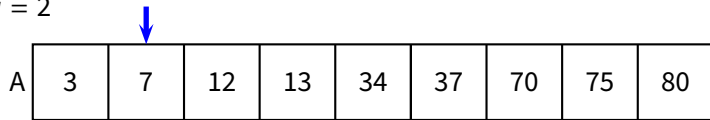
$j = 3$



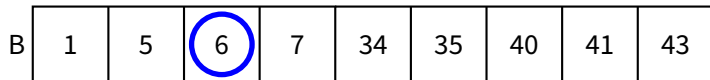
Output: 1 3 5

MERGE Algorithm

$i = 2$



A	3	7	12	13	34	37	70	75	80
---	---	---	----	----	----	----	----	----	----




B	1	5	6	7	34	35	40	41	43
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$j = 3$

Output: 1 3 5

MERGE Algorithm

$i = 2$



A	3	7	12	13	34	37	70	75	80
---	---	---	----	----	----	----	----	----	----

B	1	5	6	7	34	35	40	41	43
---	---	---	---	---	----	----	----	----	----

$j = 4$



Output: 1 3 5 6

MERGE Algorithm

$i = 2$

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---	---	---	----	----	----	----	----	----	----


B	1	5	6	7	34	35	40	41	43
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$j = 4$

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MERGE Algorithm

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---	---	---	---	---	----	----	----	----	----

$j = 5$

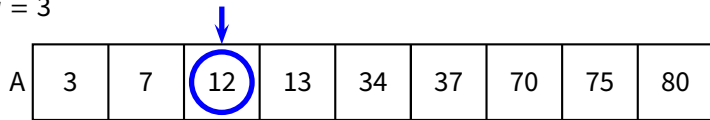


Output: 1 3 5 6 7

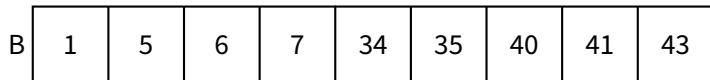
MERGE Algorithm

$i = 3$

A	3	7	12	13	34	37	70	75	80
---	---	---	----	----	----	----	----	----	----



B	1	5	6	7	34	35	40	41	43
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$j = 5$

Output: 1 3 5 6 7

MERGE Algorithm

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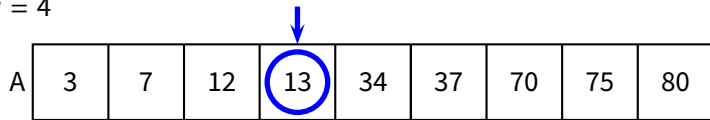


Output: 1 3 5 6 7 12

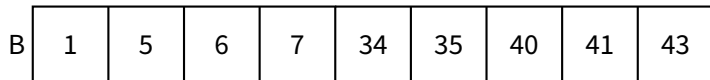
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Output: 1 3 5 6 7 12 13...

MERGE Algorithm (2)

MERGE(A, B)

```
1   $i, j = 1$ 
2   $X = \emptyset$ 
3  while  $i \leq \text{length}(A)$  or  $j \leq \text{length}(B)$ 
4      if  $i > \text{length}(A)$ 
5           $X = X \circ B[j]$            // appends  $B[j]$  to  $X$ 
6           $j = j + 1$ 
7      elseif  $j > \text{length}(B)$ 
8           $X = X \circ A[i]$ 
9           $i = i + 1$ 
10     elseif  $A[i] < B[j]$ 
11          $X = X \circ A[i]$ 
12          $i = i + 1$ 
13     else  $X = X \circ B[j]$ 
14          $j = j + 1$ 
15     return  $X$ 
```

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```

- This algorithm is incorrect! (Exercise: fix it)

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- Can we do better?

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9 **return** X

$$T(n) = \Theta(n)$$

■ Can we do better? No!

- ▶ we have to output $n = \text{length}(A) + \text{length}(B)$ elements

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 - ▶ merges two *sorted* sequences
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 - ▶ use a variant of **MERGE** that outputs *all* elements of its input sequences
 - ▶ i.e., without removing duplicates
 - ▶ assume that two parts, $A_L \circ A_R = A$, and that A_L and A_R are sorted

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 - ▶ use **MERGE** to combine A_L and A_R into a sorted sequence
 - ▶ this suggests a recursive algorithm

MERGESORT(A)

```
1  if  $length(A) == 1$ 
2      return  $A$ 
3   $m = \lfloor length(A)/2 \rfloor$ 
4   $A_L = \mathbf{MERGESORT}(A[1..m])$ 
5   $A_R = \mathbf{MERGESORT}(A[m+1..length(A)])$ 
6  return MERGE( $A_L, A_R$ )
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```

- The complexity of **MERGESORT** is

MERGESORT(*A*)

```
1  if length(A) == 1
2      return A
3  m =  $\lfloor \text{length}(\textit{A})/2 \rfloor$ 
4  AL = MERGESORT(A[1..m])
5  AR = MERGESORT(A[m + 1..length(A)])
6  return MERGE(AL, AR)
```

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$$T(n) = 2T(n/2) + O(n)$$

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$$T(n) = O(n \log n)$$

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- *General strategy*: given a problem P on input data A
 - ▶ *divide* the input A into parts A_1, A_2, \dots, A_k , usually *disjoint*, surely with $|A_i| < |A| = n$
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 - ▶ **combine** the partial solutions to obtain the solution for A
- Complexity analysis

$$T(n) = T_{\text{divide}} + \sum_{i=1}^k T(|A_i|) + T_{\text{combine}}$$

we might analyze this formula another time...

A Divide-and-Conquer Merge

MERGER(A, B)

1 **if** $length(A) == 0$

2 **return** B

3 **if** $length(B) == 0$

4 **return** A

5 **if** $A[1] < B[1]$

6 **return** $A[1] \circ \mathbf{MERGER}(A[2..length(A)], B)$

7 **else return** $B[1] \circ \mathbf{MERGER}(A, B[2..length(B)])$

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A Divide-and-Conquer Merge

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- The complexity of **MERGER** is

$$T(n) = C_1 + T(n - 1)$$

MERGER(*A*, *B*)

1 **if** *length*(*A*) == 0

2 **return** *B*

3 **if** *length*(*B*) == 0

4 **return** *A*

5 **if** *A*[1] < *B*[1]

6 **return** *A*[1] ◦ **MERGER**(*A*[2..*length*(*A*)], *B*)

7 **else return** *B*[1] ◦ **MERGER**(*A*, *B*[2..*length*(*B*)])

- Again, this algorithm is a bit incorrect (Exercise: Fix it.)
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$$T(n) = C_1 + T(n - 1) = C_1 n$$

A Divide-and-Conquer Merge

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$$T(n) = C_1 + T(n - 1) = C_1 n = O(n)$$

- Can we do better?

```
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- Again, this algorithm is a bit incorrect (Exercise: Fix it.)
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- Can we do better? No! (We knew that already)

Divide-and-Conquer Multiplication

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- Going back to multiplication...

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$$x = \boxed{X_L} \boxed{X_R} \quad \text{and} \quad y = \boxed{Y_L} \boxed{Y_R}$$

Divide-and-Conquer Multiplication

- Going back to multiplication...

$$x = \boxed{X_L} \boxed{X_R} \quad \text{and} \quad y = \boxed{Y_L} \boxed{Y_R}$$

which means $x = 2^{\ell/2}x_L + x_R$ and $y = 2^{\ell/2}y_L + y_R$, so...

$$\begin{aligned}xy &= (2^{\ell/2}x_L + x_R)(2^{\ell/2}y_L + y_R) \\ &= 2^\ell x_L y_L + 2^{\ell/2}(x_L y_R + x_R y_L) + x_R y_R\end{aligned}$$

we reduced the problem of multiplying two numbers of ℓ bits into the problem of multiplying *four* numbers of $\ell/2$ bits...

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$$T(\ell) = 4T(\ell/2) + O(\ell)$$

Divide-and-Conquer Multiplication

- Going back to multiplication...

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$$T(\ell) = 4T(\ell/2) + O(\ell)$$

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Divide-and-Conquer Multiplication (2)

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but notice that $x_L y_R + x_R y_L = (x_L + x_R)(y_R + y_L) - x_L y_L - x_R y_R$, so

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Only 3 multiplications: $x_L y_L$, $(x_L + x_R)(y_R + y_L)$, and $x_R y_R$

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Divide-and-Conquer Multiplication (2)

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which, as we will see, leads to a much better complexity

$$T(\ell) = O(\ell^{\log_2 3}) = O(\ell^{1.59})$$

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the 6th smallest element of A —a.k.a. $\text{select}(A, 6)$ —is 8

k-Smallest Element

- Idea: we split the sequence A in three parts based on a *chosen value* $v \in A$
 - ▶ A_L contains the set of elements that are *less than* v
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It is the 2nd smallest value of A_R

We use $select(A, k)$ to denote the k -smallest element of A

$$select(A, k) = \begin{cases} select(A_L, k) & \text{if } k \leq |A_L| \\ v & \text{if } |A_L| < k \leq |A_L| + |A_V| \\ select(A_R, k - |A_L| - |A_V|) & \text{if } k > |A_L| + |A_V| \end{cases}$$

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- We pick *a random element of A*

```
SELECTION( $A, k$ )
1   $v = A[\text{random}(1 \dots |A|)]$ 
2   $A_L, A_V, A_R = \emptyset$ 
3  for  $i = 1$  to  $|A|$ 
4      if  $A[i] < v$ 
5           $A_L = A_L \cup A[i]$ 
6      elseif  $A[i] == v$ 
7           $A_V = A_V \cup A[i]$ 
8      else  $A_R = A_R \cup A[i]$ 
9  if  $k \leq |A_L|$ 
10     return SELECTION( $A_L, k$ )
11 elseif  $k > |A_L| + |A_V|$ 
12     return SELECTION( $A_R, k - |A_L| - |A_V|$ )
13 else return  $v$ 
```