B-Trees

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Outline

- Search in secondary storage
- B-Trees
  - properties
  - search
  - insertion

Complexity Model

- Basic assumption so far: *data structures fit completely in main memory (RAM)*
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Disk is 10,000–100,000 times slower than RAM

Memory access/transfer	CPU cycles (≈ 1ns)
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L1 cache	4
L2 cache	10
Local L3 cache	40–75
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Local DRAM	60
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Round trip within a datacenter	500,000
HDD seek	***10,000,000***
Read 1 MB sequentially from network	10,000,000
Read 1 MB sequentially from disk	30,000,000
Round-trip time USA–Europe	150,000,000

Modeling Disk Access

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  DISK-READ(x) reads the object into memory, allowing us to refer to it (and modify it) through x
- Any changes to the object in memory must be eventually saved onto the disk

  DISK-WRITE(x) writes the object onto the disk (if the object was modified)

Binary Trees on Disk

- Assume each node *x* is stored on disk

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ITERATIVE-TREE-SEARCH(T, k)
x = T.root
while x ≠ NIL
        DISK-READ(x)
        if k == x.key
                return x
        elseif k < x.key
                x = x.left
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cost

Binary Trees on Disk

- Assume each node x is stored on disk

ITERATIVE-TREE-SEARCH(T, k)	*cost*
$x = T.root$	c
while $x \neq$ NIL	c
DISK-READ(x)	$100000c$
if $k == x.key$	c
return x	c
elseif $k < x.key$	c
$x = x.left$	c
else $x = x.right$	c
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- Assume we store the nodes of a search tree on disk
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- Rationale
  - basic in-memory operations are much cheaper
  - the bottleneck is with node accesses, which involve **Disk-Read** and **Disk-Write** operations

Idea

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- In a balanced *binary* tree, n keys require a tree of height $h = \lfloor \log_2 n \rfloor$
  - all the important operations require access to $O(h)$ nodes
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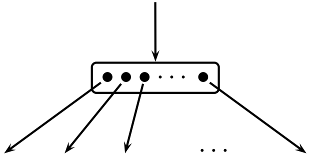
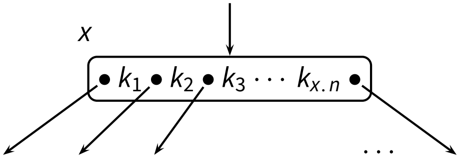
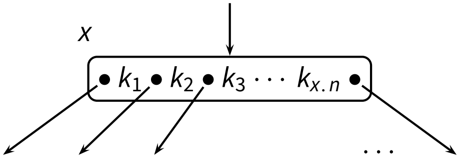


E.g., if $d = 1000$, then ***only three accesses*** ($h = 2$) cover ***up to one billion keys***


x
k1 k2 k3 · · · kx.n
. . .



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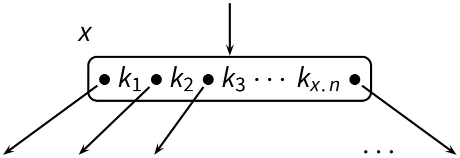
  - $x.n$ is the number of keys stored at each node



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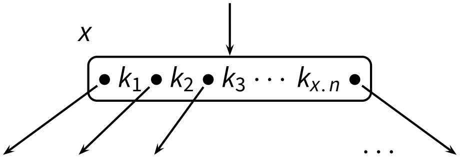

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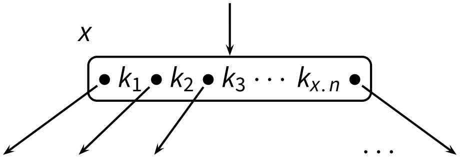
  - $x.leaf$ is a Boolean flag that is TRUE if x is a *leaf node* or FALSE if x is an *internal node*

Definition of a B-Tree



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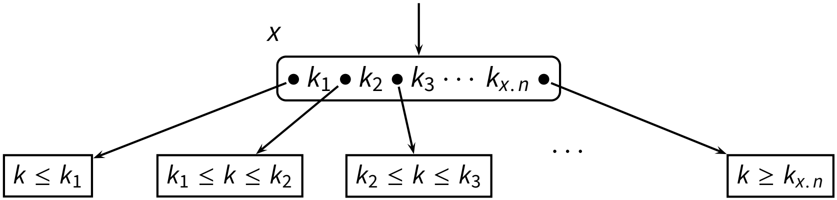
  - $x.leaf$ is a Boolean flag that is TRUE if x is a *leaf node* or FALSE if x is an *internal node*
  - $x.c[1], x.c[2], \ldots, x.c[x.n+1]$ are the $x.n+1$ pointers to its children, if x is an *internal node*

Definition of a B-Tree (2)



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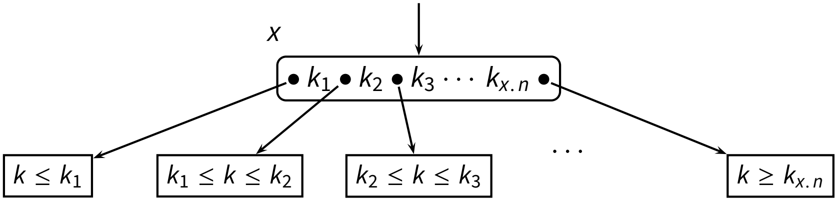

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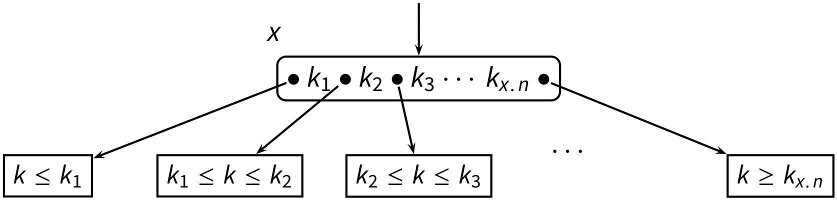

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  $x.c[1] \longrightarrow$ subtree containing keys $k \leq x.key[1]$

  $x.c[2] \longrightarrow$ subtree containing keys $k, x.key[1] \leq k \leq x.key[2]$

  $x.c[3] \longrightarrow$ subtree containing keys $k, x.key[2] \leq k \leq x.key[3]$

  $\ldots$

  $x.c[x.n+1] \longrightarrow$ subtree containing keys $k, k \geq x.key[x.n]$

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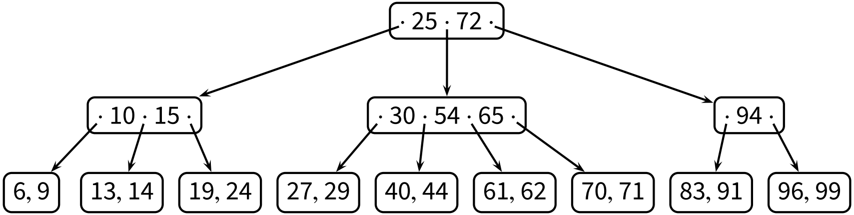
- Let $t \geq 2$ be the ***minimum degree*** of the B-tree
  - every node other than the root must have ***at least*** $t - 1$ ***keys***
  - every node must contain ***at most*** $2t - 1$ ***keys***
    - a node is *full* when it contains exactly $2t - 1$ keys
    - a full node has $2t$ children

Example



· 25 · 72 ·
· 10 · 15 ·
· 30 · 54 · 65 ·
· 94 ·
6, 9
13, 14
19, 24
27, 29
40, 44
61, 62
70, 71
83, 91
96, 99

Search in B-Trees

```
B-TREE-SEARCH(x, k)
i = 1
while i ≤ x.n and k > x.key[i]
        i = i + 1
if i ≤ x.n and k == x.key[i]
        return (x, i)
if x.leaf
        return NIL
else DISK-READ(x.c[i])
        return B-TREE-SEARCH(x.c[i], k)
```

Height of a B-Tree

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- each subtree contains $1 + t + t^2 \cdots + t^{h-1}$ nodes, each one containing $t-1$ keys, so

$$n \geq 1 + 2(t^h - 1)$$

Splitting

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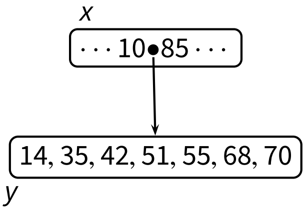

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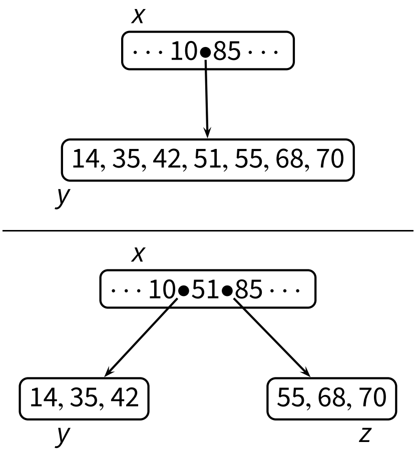

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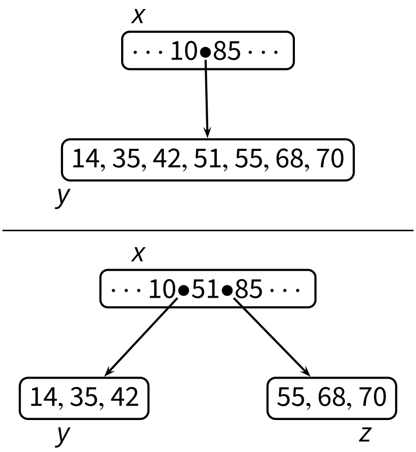


```
B-TREE-SPLIT-CHILD(x, i, y)
z = ALLOCATE-NODE()
z.leaf = y.leaf
z.n = t - 1
for j = 1 to t - 1
    z.key[j] = y.key[j + t]
if not y.leaf
    for j = 1 to t
        z.c[j] = y.c[j + t]
y.n = t - 1
for j = x.n + 1 downto i + 1
    x.c[j + 1] = x.c[j]
for j = x.n downto i
    x.key[j + 1] = x.key[j]
x.key[i] = y.key[t]
x.n = x.n + 1
DISK-WRITE(y)
DISK-WRITE(z)
DISK-WRITE(x)
```

Complexity of B-TREE-SPLIT-CHILD

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- $\Theta(t)$ basic CPU operations

Complexity of B-Tree-Split-Child

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- $\Theta(t)$ basic CPU operations
- 3 **Disk-Write** operations

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    z = ALLOCATE-NODE()
    z.leaf = y.leaf
    z.n = t − 1
    for j = 1 to t − 1
        x.key[j] = x.key[j + t]
    if not x.leaf
        for j = 1 to t
            z.c[j] = y.c[j + t]
    y.n = t − 1
    for j = x.n + 1 downto i + 1
        x.c[j + 1] = x.c[j]
    for j = x.n downto i
        x.key[j + 1] = x.key[j]
    x.key[i] = y.key[t]
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    DISK-WRITE(y)
    DISK-WRITE(z)
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Insertion Under Non-Full Node

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```
B-TREE-INSERT-NONFULL(x, k)
i = x.n                                  // assume x is not full
if x.leaf
    while i ≥ 1 and k < x.key[i]
        x.key[i + 1] = x.key[i]
        i = i − 1
    x.key[i + 1] = k
    x.n = x.n + 1
    DISK-WRITE(x)
else while i ≥ 1 and k < x.key[i]
        i = i − 1
    i = i + 1
    DISK-READ(x.c[i])
    if x.c[i].n == 2t − 1                // child x.c[i] is full
        B-TREE-SPLIT-CHILD(x, i, x.c[i])
        if k > x.key[i]
            i = i + 1
    B-TREE-INSERT-NONFULL(x.c[i], k)
```

Insertion Procedure

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```
B-TREE-INSERT(T, k)
r = T.root
if r.n == 2t − 1
    s = ALLOCATE-NODE()
    T.root = s
    s.leaf = FALSE
    s.n = 0
    s.c[1] = r
    B-TREE-SPLIT-CHILD(s, 1, r)
    B-TREE-INSERT-NONFULL(s, k)
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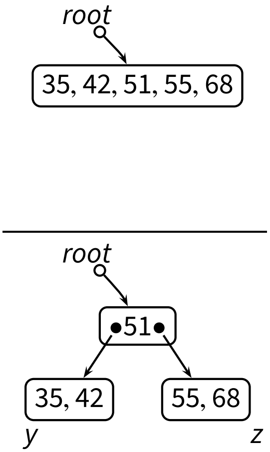

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- $O(h) = O(\log_t n)$ disk-access operations

- What is the complexity of **B-Tree-Insert**?
- $O(th) = O(t \log_t n)$ basic CPU steps operations
- $O(h) = O(\log_t n)$ disk-access operations
- The best value for t can be determined according to
  - the ratio between CPU (RAM) speed and disk-access time
  - the *block-size* of the disk, which determines the maximum size of an object that can be accessed (read/write) in one shot