Binary Search Trees

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Outline

- Binary search trees
- Randomized binary search trees

- A ***binary search tree*** implements a *dynamic set*
  - over a ***totally ordered domain***

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  - *iteration:* **TREE-SUCCESSOR**(x) and **TREE-PREDECESSOR**(x) find the successor and predecessor, respectively, of an element x

- *Implementation*
  - *T* represents the tree, which consists of a set of ***nodes***

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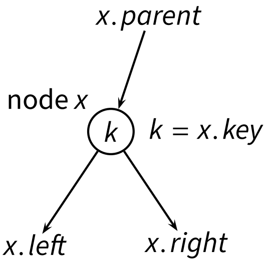
Node x

- $x.parent$ is the parent of node x
- $x.key$ is the key stored in node x
- $x.left$ is the left child of node x
- $x.right$ is the right child of node x



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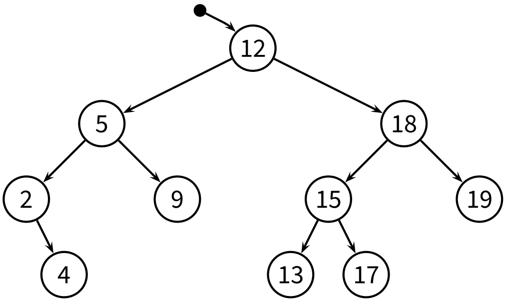

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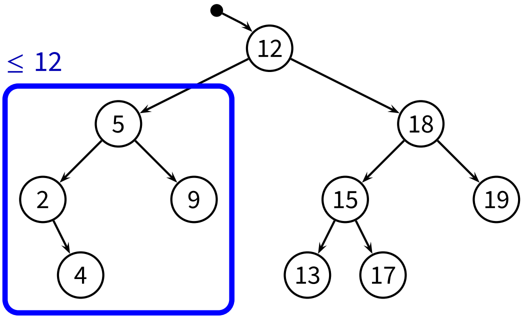

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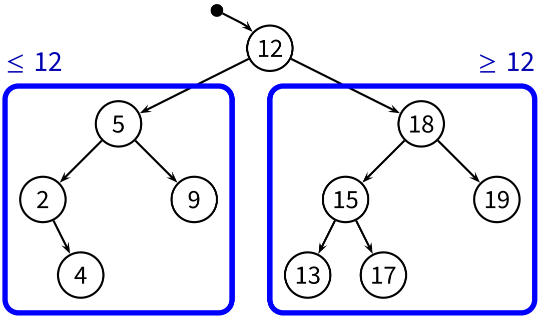

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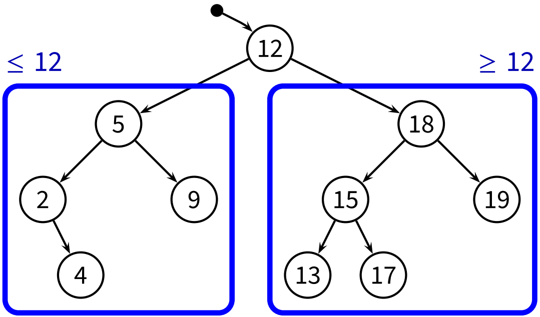

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  - for all nodes x, y, and z
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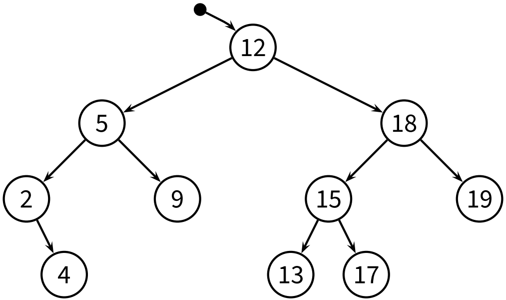

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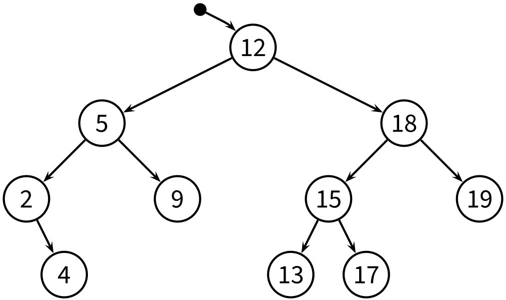

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INORDER-TREE-WALK(x)
if x ≠ NIL
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And then we need a “starter” procedure

```
INORDER-TREE-WALK-START(T)
INORDER-TREE-WALK(T.root)
```

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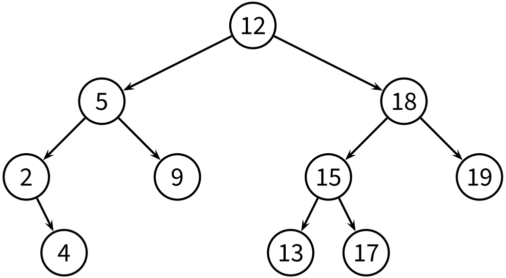

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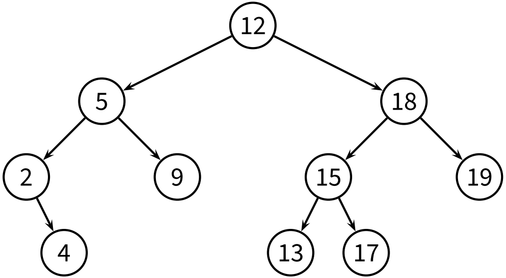

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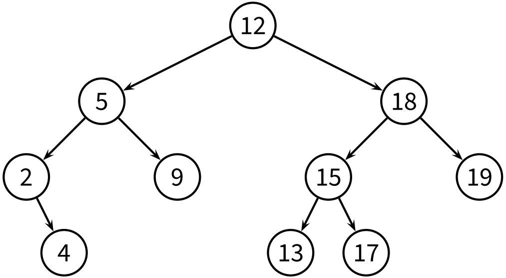

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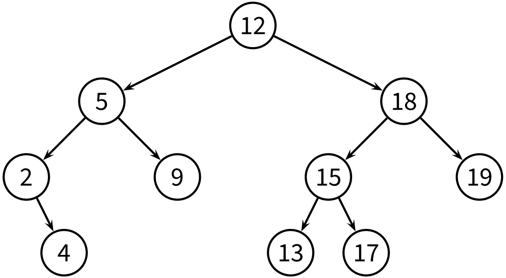

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REVERSE-ORDER-TREE-WALK(x)
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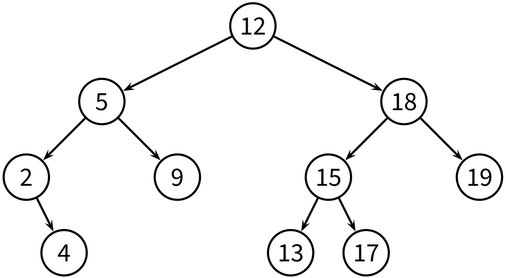

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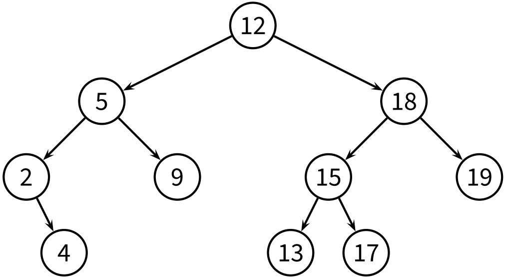

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We could prove this using the *substitution method*

- Can we do better? No!
  - the length of the output is $\Theta(n)$

Minimum and Maximum Keys

- Recall the *binary-search-tree property*
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- So, the minimum key is in all the way to the left
  - similarly, the maximum key is all the way to the right

```
TREE-MINIMUM(x)
while x.left ≠ NIL
    x = x.left
return x
```

```
TREE-MAXIMUM(x)
while x.right ≠ NIL
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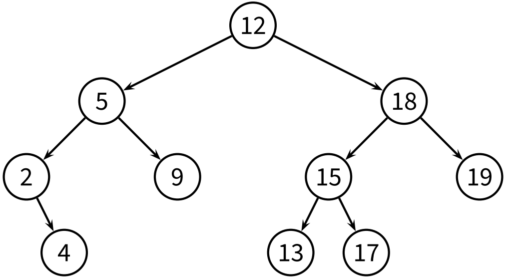

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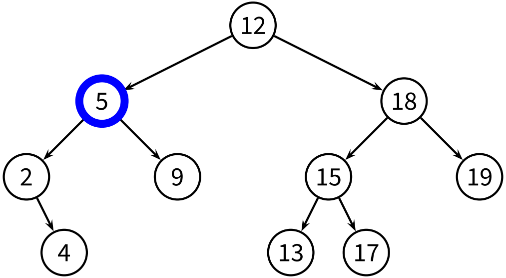

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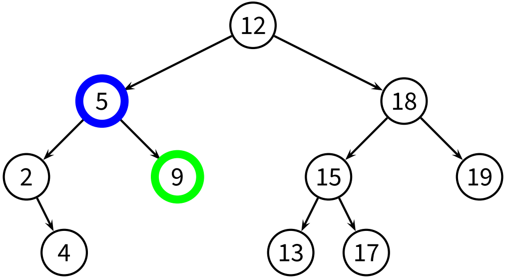

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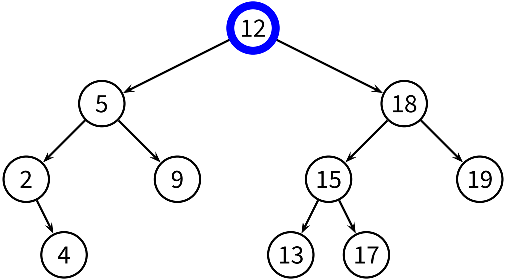

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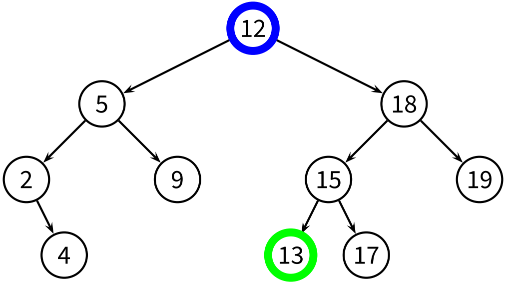

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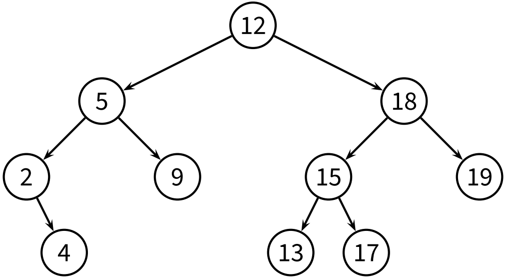

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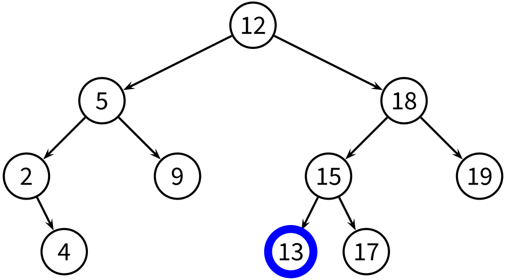

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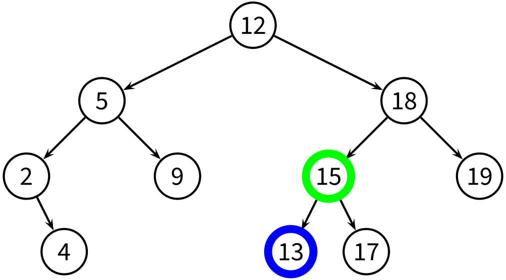

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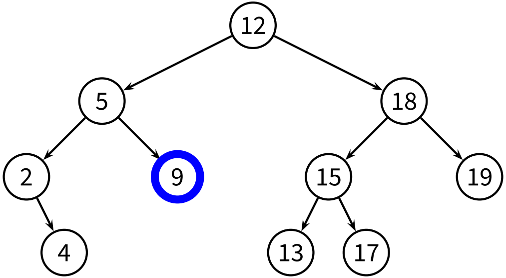

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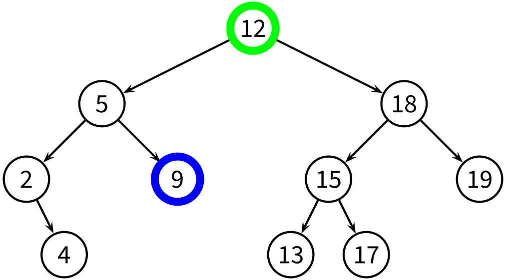

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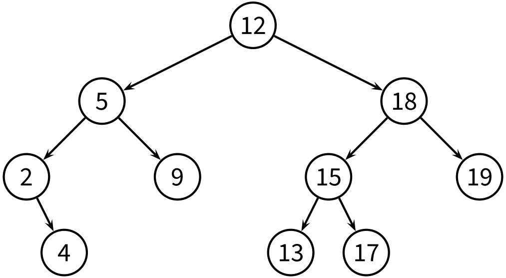

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- Otherwise it is the *first ancestor a* of *x* such that *x* falls in the *left* subtree of *a*

Successor and Predecessor(2)

```
TREE-SUCCESSOR(x)
if x.right ≠ NIL
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while y ≠ NIL and x = y.right
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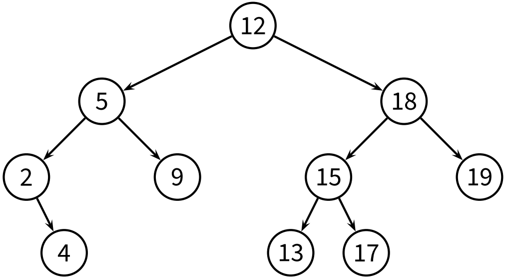

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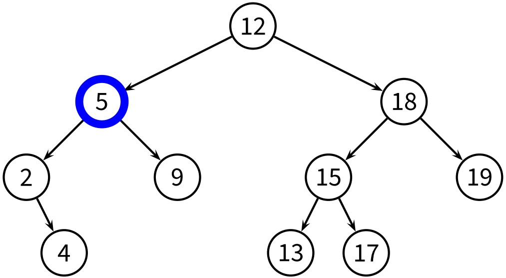

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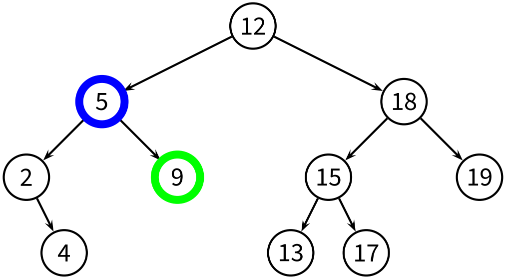

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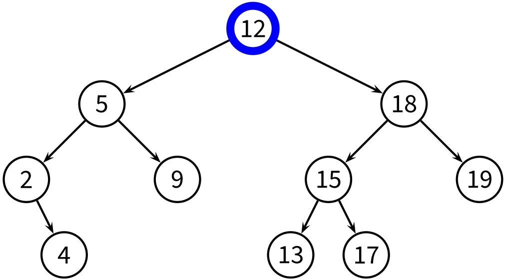

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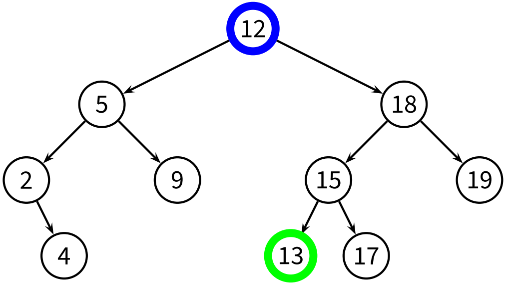

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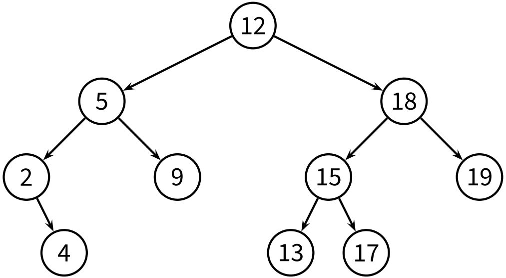

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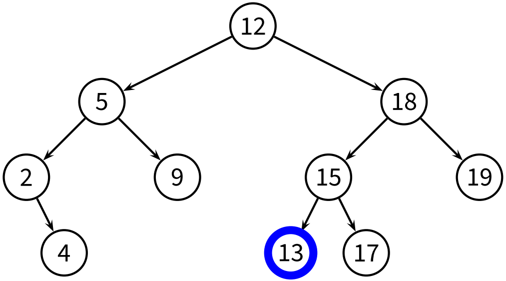

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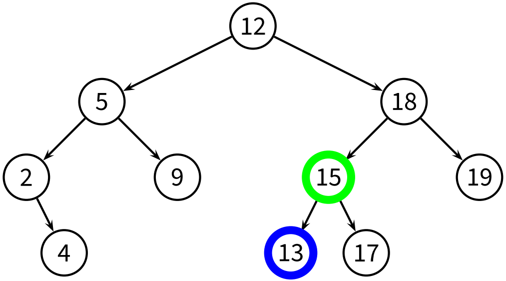

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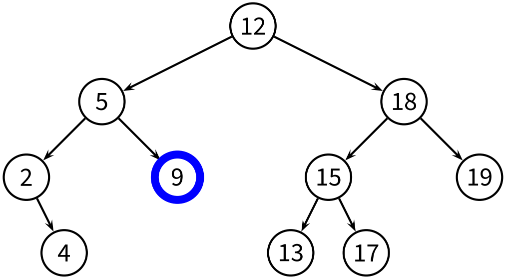

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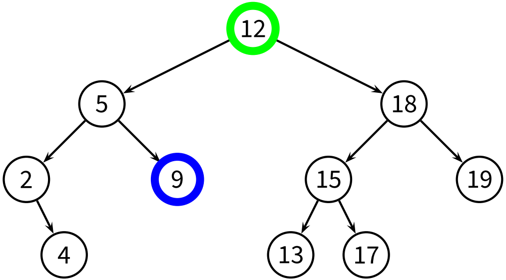

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TREE-SEARCH(x, k)
    if x = NIL or k = x.key
        return x
    if k < x.key
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- Is this correct?

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```
TREE-SEARCH(x, k)
1  if x = NIL or k = x.key
2          return x
3  if k < x.key
4          return TREE-SEARCH(x.left, k)
5  else return TREE-SEARCH(x.right, k)
```

- Is this correct? Yes, thanks to the *binary-search-tree property*

Search

- *Binary search* (thus the name of the tree)

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TREE-SEARCH(x, k)
if x = NIL or k = x.key
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- Complexity?

$$T(n) = \Theta(\textit{depth of the tree})$$

$$T(n) = O(n)$$

- Iterative *binary search*

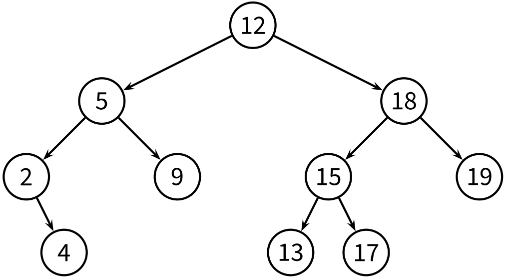
Search (2)

- Iterative *binary search*

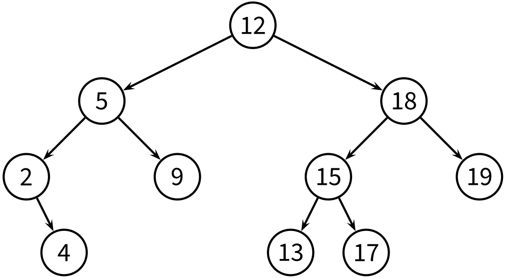
```
ITERATIVE-TREE-SEARCH(T, k)
x = T.root
while x ≠ NIL ∧ k ≠ x.key
    if k < x.key
        x = x.left
    else x = x.right
return x
```

Insertion

Insertion



Insertion



- *Idea*
  - in order to insert x, we *search* for x (more precisely $x.key$)
  - if we don't find it, we add it where the search stopped

Insertion (2)

```
TREE-INSERT(T, z)
y = NIL
x = T.root
while x ≠ NIL
    y = x
    if z.key < x.key
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    else x = x.right
z.parent = y
if y = NIL
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    y.left = z
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Insertion (2)

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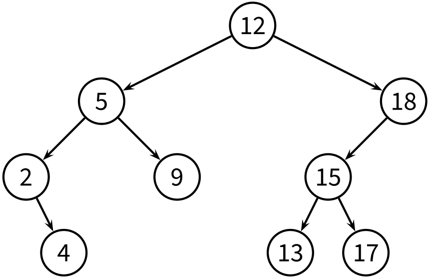

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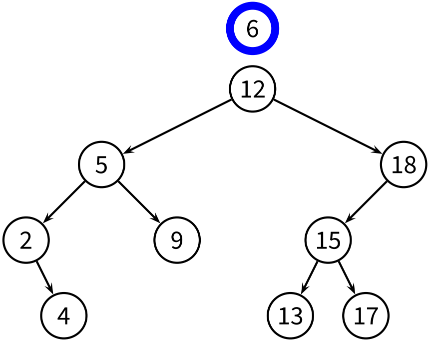

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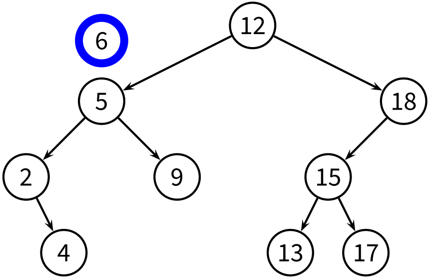

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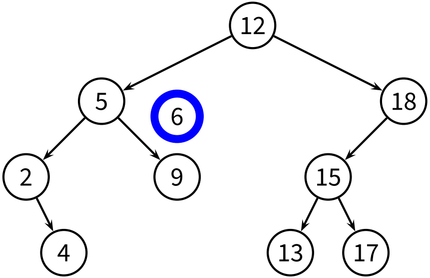

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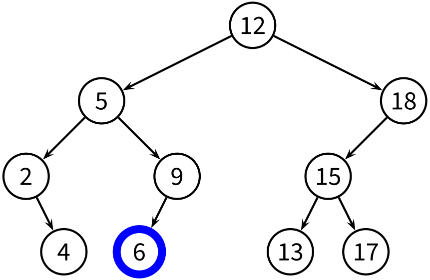

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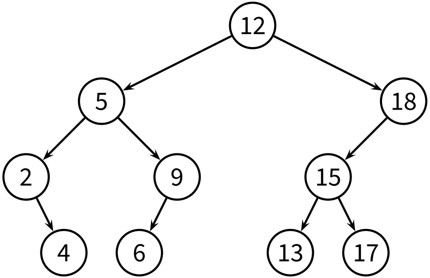

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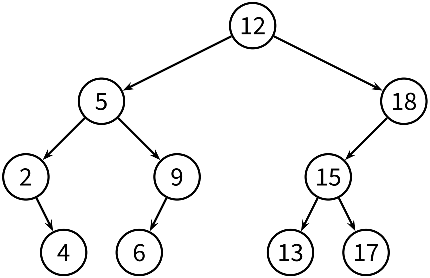
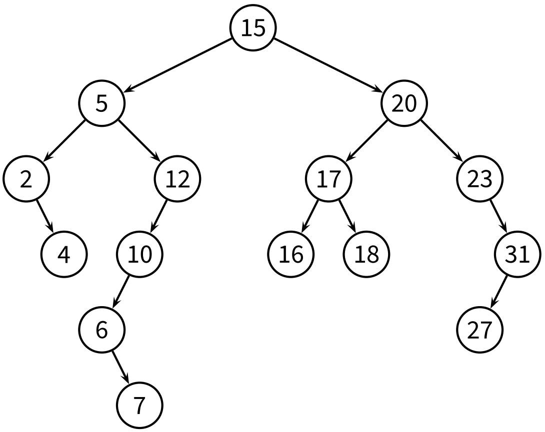
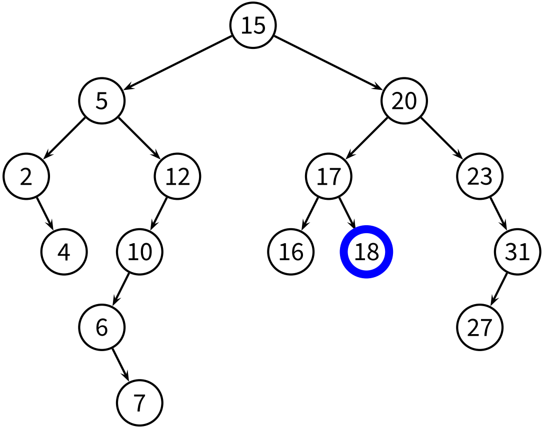


$$T(n) = \Theta(h)$$

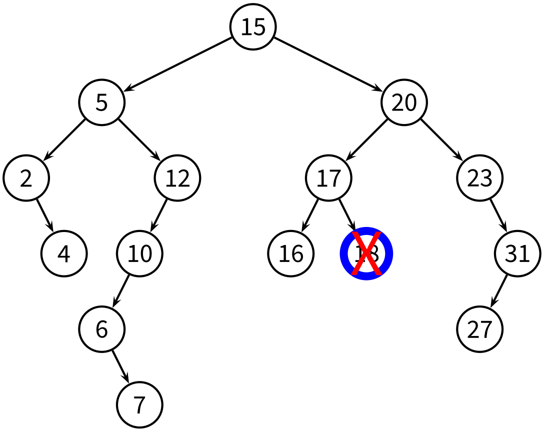
Deletion

Deletion





1. z has no children



1. *z* has no children
   - simply remove *z*

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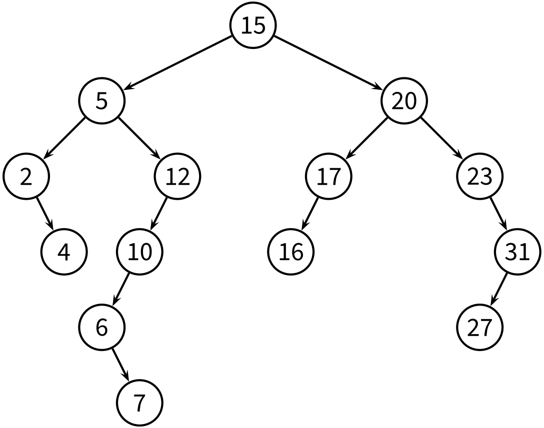

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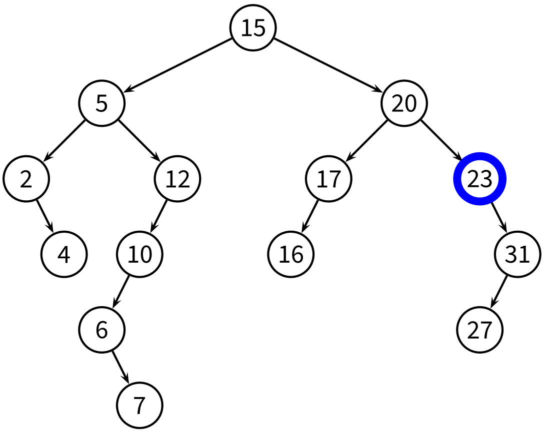

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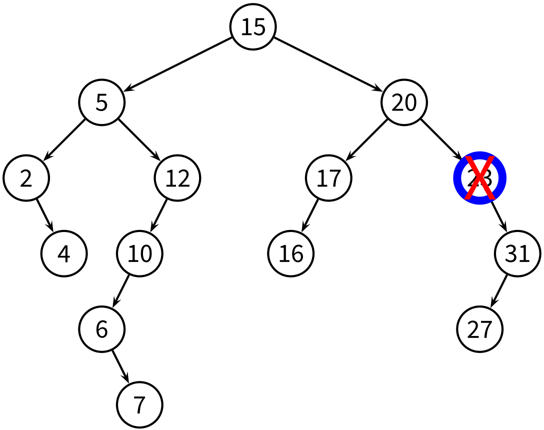

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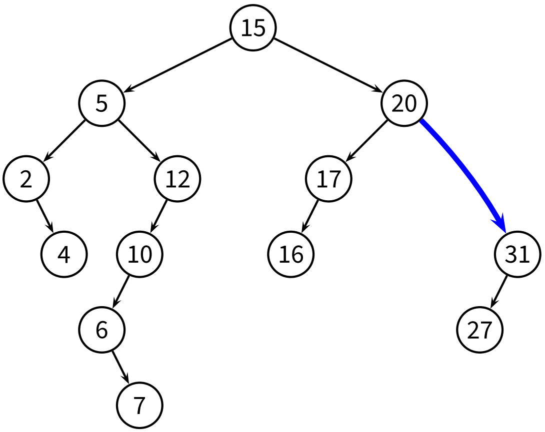

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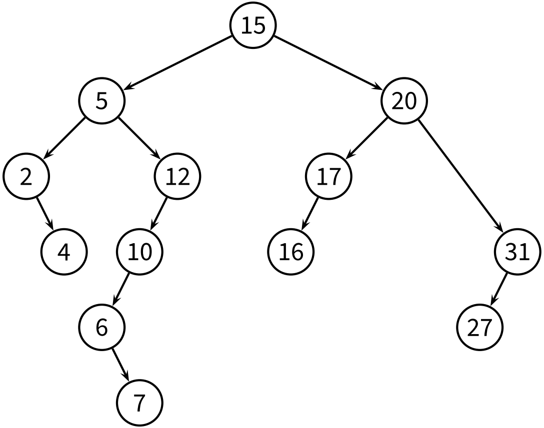
   - connect $z.parent$ to $z.right$

Deletion



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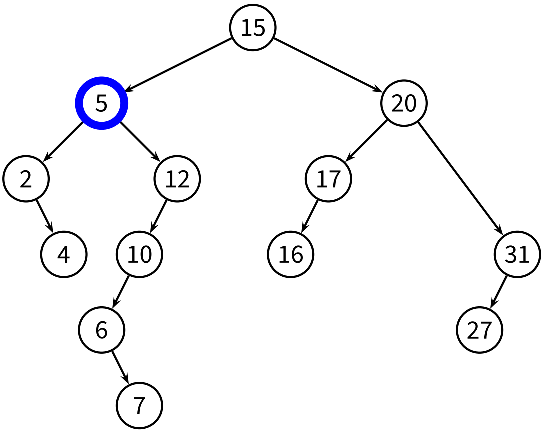
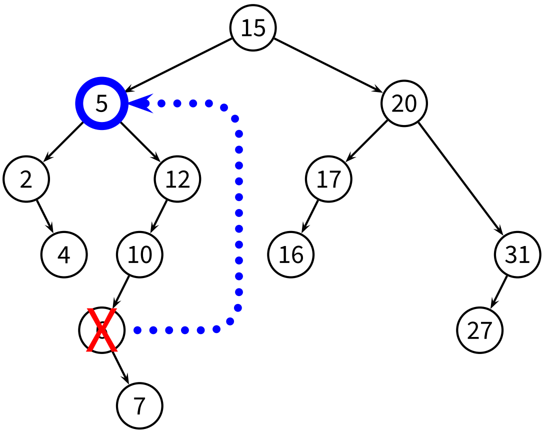


1. *z* has no children
   - simply remove *z*
2. *z* has one child
   - remove *z*
   - connect $z.parent$ to $z.right$
3. *z* has two children



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3. *z* has two children
   - replace *z* with $y = $ **Tree-Successor**(z)
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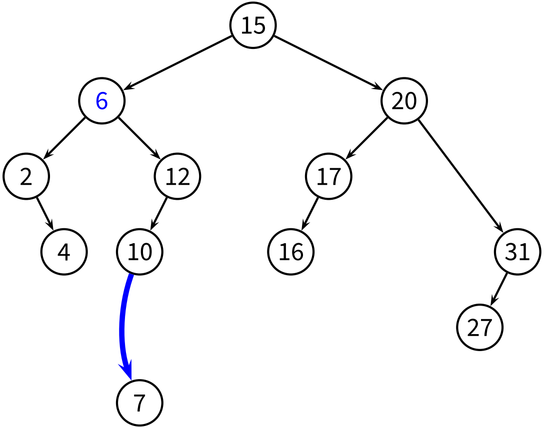

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   - replace z with $y = \textbf{Tree-Successor}(z)$
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Deletion (2)

```
TREE-DELETE(T, z)
if z.left = NIL or z.right = NIL
        y = z
else y = TREE-SUCCESSOR(z)
if y.left ≠ NIL
        x = y.left
else x = y.right
if x ≠ NIL
        x.parent = y.parent
if y.parent == NIL
        T.root = x
else if y = y.parent.left
                y.parent.left = x
        else y.parent.right = x
if y ≠ z
        z.key = y.key
        copy any other data from y into z
```

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- Insertion, search, and deletion operations have complexity $\Theta(h)$, where h is the height of the tree

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  - i.e., with ordered sequences
  - the problem is that the "worst" case is not that uncommon
- ***Idea:*** use randomization to turn all cases into the average case

Randomized Insertion

- ***Idea 1:*** insert every sequence as a random sequence

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- ***Idea 2:*** we can obtain a random permutation of the input sequence by randomly alternating two insertion procedures
  - *tail insertion:* this is what **Tree-Insert** does
  - *head insertion:* for this we need a new procedure **Tree-Root-Insert**
    - inserts n in T as if n was inserted as the first element

Randomized Insertion (2)

```
TREE-RANDOMIZED-INSERT1(T, z)
r = uniformly random value from {1, ..., t.size + 1}
if r = 1
        TREE-ROOT-INSERT(T, z)
else TREE-INSERT(T, z)
```

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    - no, clearly the last element can only go to the top or to the bottom
- It is true that any node has the same probability of being inserted at the top
    - this suggests a recursive application of this same procedure

Randomized Insertion (3)

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```
TREE-RANDOMIZED-INSERT(t, z)
if t = NIL
        return z
r = uniformly random value from {1, . . . , t.size + 1}
if r = 1                                    // Pr[r = 1] = 1/(t.size + 1)
        z.size = t.size + 1
        return TREE-ROOT-INSERT(t, z)
if z.key < t.key
        t.left = TREE-RANDOMIZED-INSERT(t.left, z)
else t.right = TREE-RANDOMIZED-INSERT(t.right, z)
t.size = t.size + 1
return t
```

Randomized Insertion (3)

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if r = 1                              // Pr[r = 1] = 1/(t.size + 1)
        z.size = t.size + 1
        return TREE-ROOT-INSERT(t, z)
if z.key < t.key
        t.left = TREE-RANDOMIZED-INSERT(t.left, z)
else t.right = TREE-RANDOMIZED-INSERT(t.right, z)
t.size = t.size + 1
return t
```
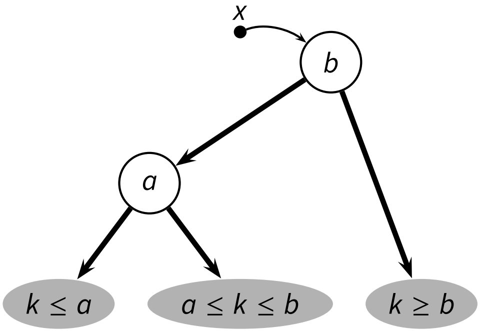
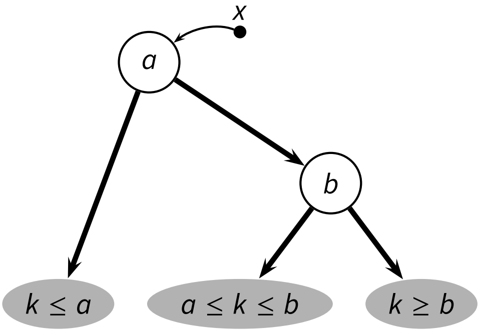

- Looks like this one really simulates a random permutation…

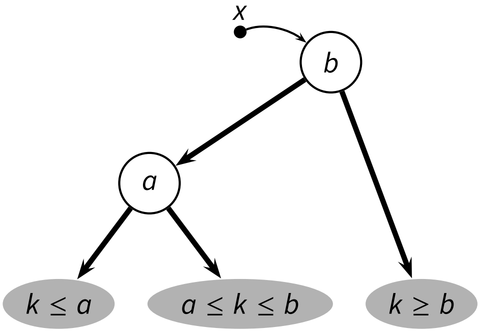
Rotation

Rotation





- x = **Right-Rotate**(x)



- x = **Right-Rotate**(x)
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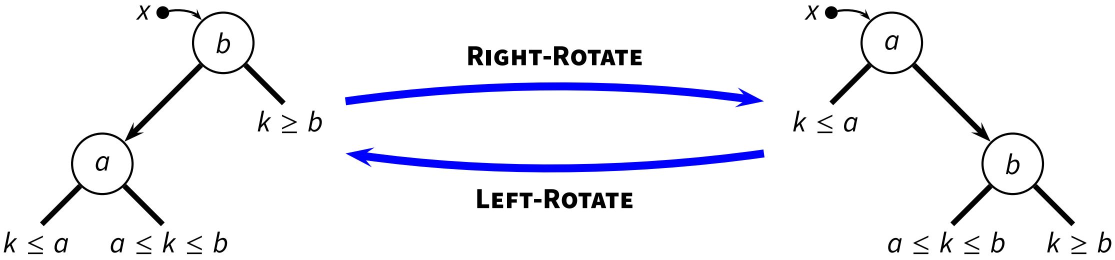
Rotation

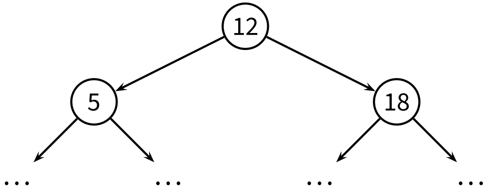


RIGHT-ROTATE(x)

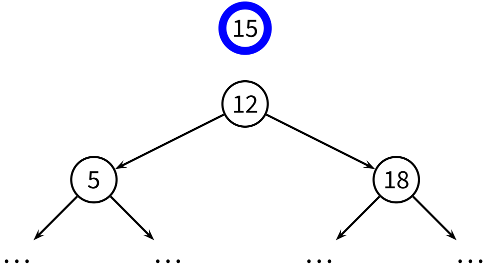
```
l = x.left
x.left = l.right
l.right = x
return l
```

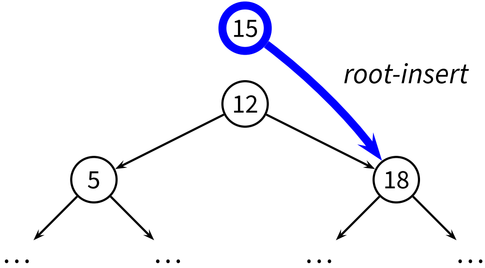
LEFT-ROTATE(x)

```
r = x.right
x.right = r.left
r.left = x
return r
```

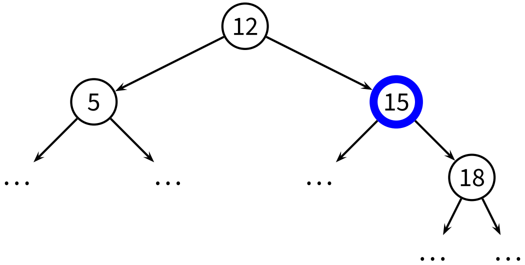

12
5
18
...
...
...
...

Root Insertion





1. Recursively insert z at the root of the appropriate subtree (right)



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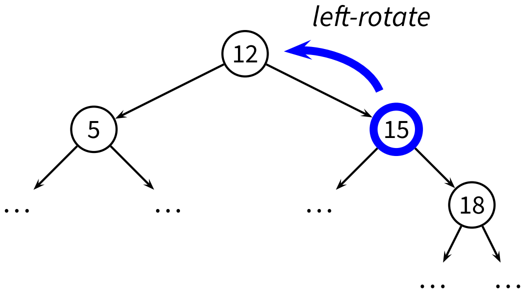

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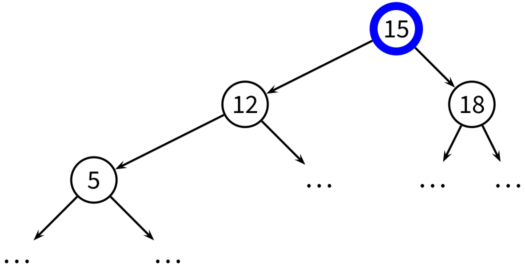
2. Rotate x with z (left-rotate)



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2. Rotate x with z (left-rotate)

Root Insertion (2)

```
TREE-ROOT-INSERT(x, z)
if x = NIL
        return z
if z.key < x.key
        x.left = TREE-ROOT-INSERT(x.left, z)
        return RIGHT-ROTATE(x)
else x.right = TREE-ROOT-INSERT(x.right, z)
        return LEFT-ROTATE(x)
```

Observation

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  - *amortized maintenance:* e.g., balancing a BST or resizing a hash table
    - relatively expensive but “amortized” operations
  - *optimized data structures:* a self-balanced data structure
    - guaranteed $O(\log n)$ complexity bounds