Analysis of Insertion Sort

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Outline

- Sorting
- Insertion Sort
- Analysis

- **Input:** a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$

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$$A = \boxed{6}\boxed{8}\boxed{3}\boxed{2}\boxed{7}\boxed{6}\boxed{11}\boxed{5}\boxed{9}\boxed{4}$$

Sorting

- **Input:** a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$

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- Typically, A is implemented as an array

in-place sort

$$A = \boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{6}\boxed{7}\boxed{8}\boxed{9}\boxed{11}$$

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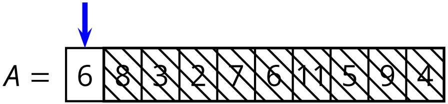

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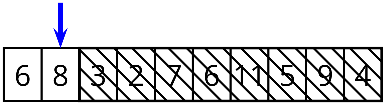

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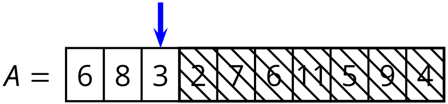

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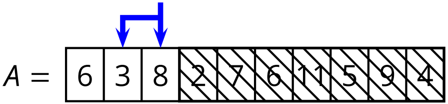

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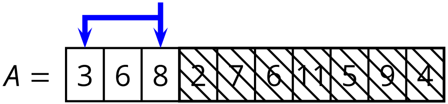

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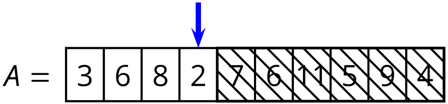

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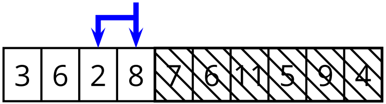

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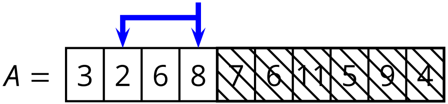

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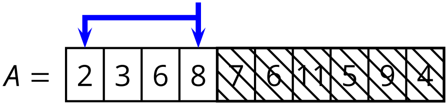

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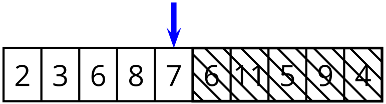
$A =$ 

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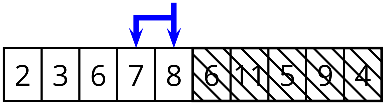

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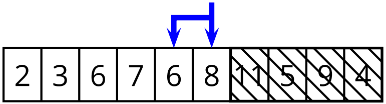

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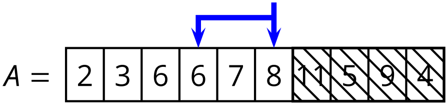

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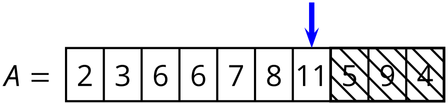

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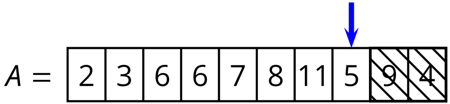

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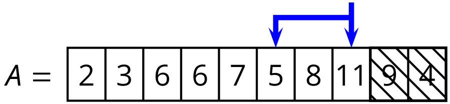

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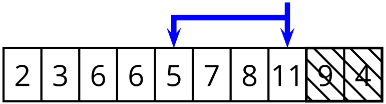

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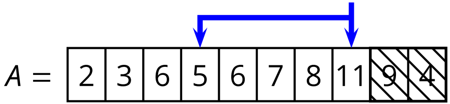

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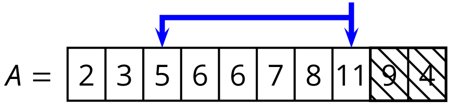

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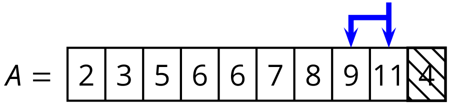

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Insertion Sort (2)

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
        j = j − 1
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- Is **INSERTION-SORT** *correct?*
- What is the time complexity of **INSERTION-SORT**?
- Can we do better?

Complexity of INSERTION-SORT

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Complexity of INSERTION-SORT

```
INSERTION-SORT(A)
1  for i = 2 to length(A)
2      j = i
3      while j > 1 and A[j − 1] > A[j]
4          swap A[j] and A[j − 1]
5          j = j − 1
```

- Outer loop (lines 1–5) runs exactly $n - 1$ times (with $n = \mathit{length}(A)$)
- What about the inner loop (lines 3–5)?
  - best, worst, and average case?

Complexity of INSERTION-SORT (2)

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- **Best case:**

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  - what case is this?

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- **Worst case:**

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- **Best case:** the inner loop is *never* executed
  - what case is this?

- **Worst case:** the inner loop is executed exactly $j - 1$ times for every iteration of the outer loop
  - what case is this?

Complexity of INSERTION-SORT (3)

- The worst-case complexity is when the inner loop is executed exactly $j - 1$ times, so

$$T(n) = \sum_{j=2}^{n} (j - 1)$$

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$T(n)$ is the *arithmetic series* $\sum_{k=1}^{n-1} k$, so

$$T(n) = \frac{n(n-1)}{2}$$

$$\boxed{T(n) = \Theta(n^2)}$$

Complexity of Insertion-Sort (3)

- The worst-case complexity is when the inner loop is executed exactly $j - 1$ times, so

$$T(n) = \sum_{j=2}^{n} (j - 1)$$

  $T(n)$ is the *arithmetic series* $\sum_{k=1}^{n-1} k$, so

$$T(n) = \frac{n(n-1)}{2}$$

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Correctness

- Does **INSERTION-SORT** terminate for all valid inputs?

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- If so, does it satisfy the conditions of the sorting problem?
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  - A is *sorted:* $A[1] \leq A[2] \leq \cdots \leq A[\mathit{length}(A)]$

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- We want ***a formal proof of correctness***
  - does not seem straightforward...

The Logic of Algorithmic Steps

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// must return a value greater than n
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Example 2: (branching)

```
SORTTWO(A)
// must sort (in-place) an array of 2 elements
if A[1] > A[2]
    t = A[1]
    A[1] = A[2]
    A[2] = t
```

Loop Invariants

- We formulate a *loop-invariant* condition C
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  - C is relevant to the problem definition: we use C at the end of a loop to prove the correctness of the result
- Then, we only need to prove that the algorithm terminates

Loop Invariants (2)

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- Proof of validity (i.e., that *C* is indeed a loop invariant): typical *proof by induction*
  - ***initialization:*** we must prove that
    the invariant C is true before entering the loop
  - ***maintenance:*** we must prove that
    if *C is true at the beginning of a cycle* ***then*** *it remains true after one cycle*

Loop Invariant for INSERTION-SORT

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
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- The main idea is to insert $A[i]$ in $A[1 \mathinner{.\,.} i-1]$ so as to maintain a *sorted subsequence* $A[1 \mathinner{.\,.} i]$

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- The main idea is to insert $A[i]$ in $A[1 \,..\, i-1]$ so as to maintain a *sorted subsequence* $A[1 \,..\, i]$

- ***Invariant:*** (outer loop) *the subarray* $A[1 \,..\, i-1]$ *consists of the elements originally in* $A[1 \,..\, i-1]$ *in sorted order*

Loop Invariant for INSERTION-SORT (2)

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- **Initialization:** $j = 2$, so $A[1 \mathinner{..} j-1]$ is the single element $A[1]$
  - $A[1]$ contains the original element in $A[1]$
  - $A[1]$ is trivially sorted

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- **Maintenance:** informally, if $A[1 \mathinner{..} i-1]$ is a permutation of the original $A[1 \mathinner{..} i-1]$ and $A[1 \mathinner{..} i-1]$ is sorted (invariant), then *if* we enter the inner loop:
  - shifts the subarray $A[k \mathinner{..} i-1]$ by one position to the right
  - inserts *key*, which was originally in $A[i]$ at its proper position $1 \le k \le i-1$, in sorted order

Loop Invariant for INSERTION-SORT (4)

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
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- **Termination: INSERTION-SORT** terminates with $i = length(A) + 1$; the invariant states that
  - $A[1 \,..\, i-1]$ is a permutation of the original $A[1 \ldots i-1]$
  - $A[1 \,..\, i-1]$ is sorted

Given the termination condition, $A[1 \,..\, i-1]$ is the whole A

So **INSERTION-SORT** is *correct!*

- You are given a problem P and an algorithm A
  - P formally defines a *correctness* condition
  - assume, for simplicity, that A consists of one loop

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5. Prove that $X \wedge C \Rightarrow P$, which means that A is correct

Exercise: Analyze Selection-Sort

```
SELECTION-SORT(A)
n = length(A)
for i = 1 to n − 1
    smallest = i
    for j = i + 1 to n
        if A[j] < A[smallest]
            smallest = j
    swap A[i] and A[smallest]
```

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- Complexity?
  - worst, best, and average case?

Exercise: Analyze Bubblesort

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BUBBLESORT(A)
for i = 1 to length(A)
    for j = length(A) downto i + 1
        if A[j] < A[j − 1]
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Exercise: Analyze Bubblesort

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