Graphs: Representation and Elementary Algorithms

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Università della Svizzera italiana

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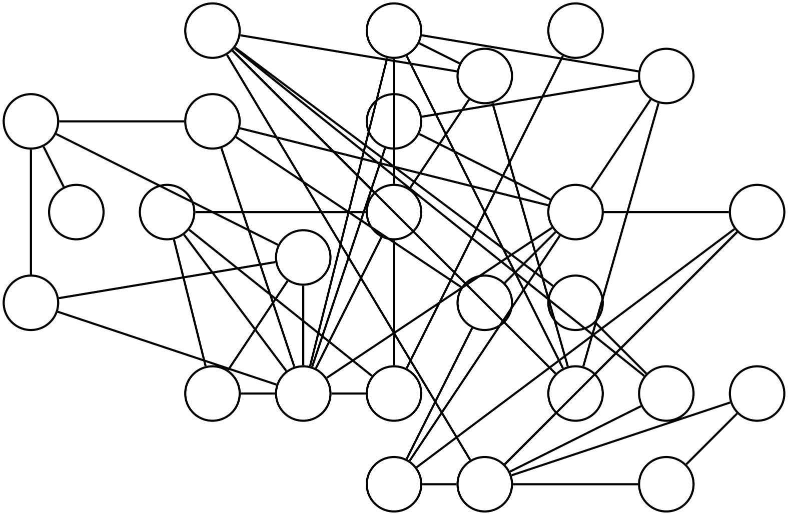
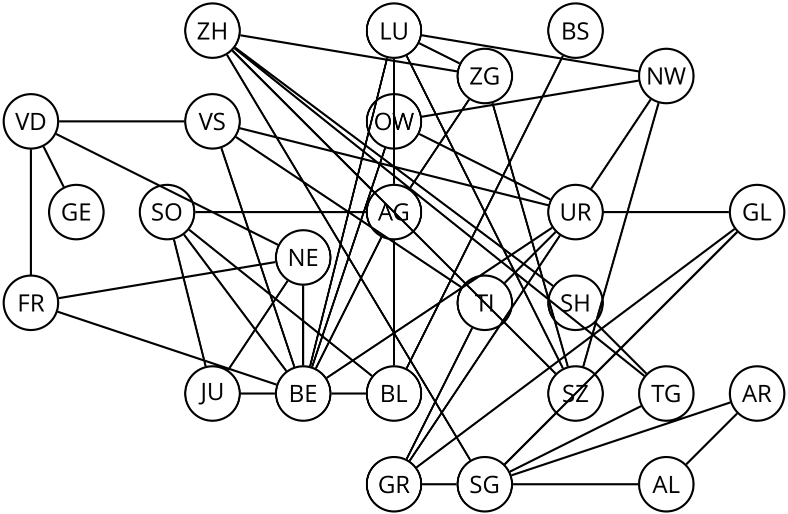
Outline

- Graphs: definitions
- Representations
- Breadth-first search
- Depth-first search

Example



Example



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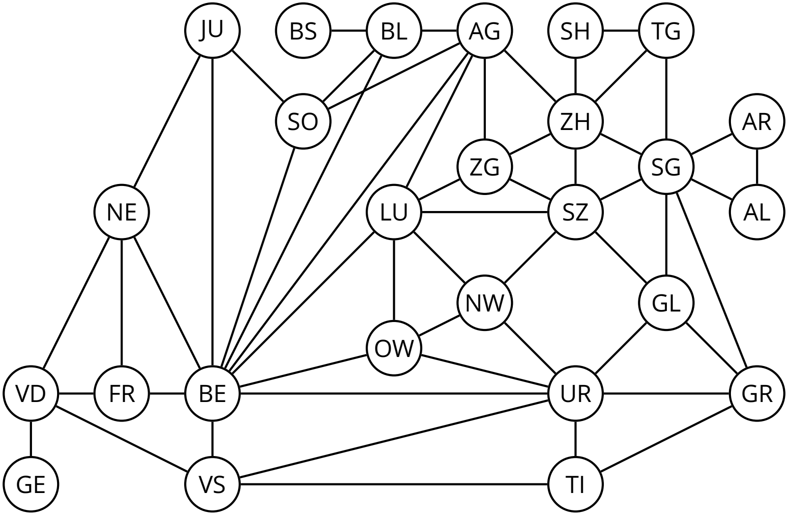
Same Example (Better Layout)



Many Models and Applications

- Social networks: *who knows who*
- The Web graph: *which page links to which*
- The Internet graph: *which router links to which*
- Citation graphs: *who references whose papers*
- Planar graphs: *which country is next to which*
- Well-shaped meshes: *pretty pictures with triangles*
- Geometric graphs: *who is near who*
- Random graphs: *whichever…*

Examples and descriptions taken from Daniel A. Spielman's course "Graphs and Networks."

- A ***graph***

$$G = (V, E)$$

- V is the set of ***vertices*** (also called ***nodes***)
- E is the set of ***edges***

Definitions

- A ***graph***

$$G = (V, E)$$

- V is the set of ***vertices*** (also called ***nodes***)

- E is the set of ***edges***
  - $E \subseteq V \times V$, i.e., E is a ***relation between vertices***
  - an edge $e = (u, v) \in V$ is a pair of vertices $u \in V$ and $v \in V$

- A ***graph***

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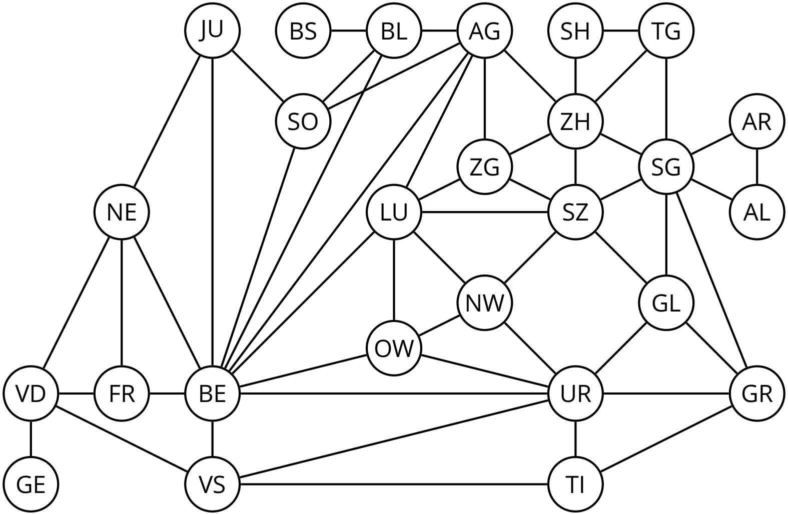
- An *undirected* graph is characterized by a *symmetric* relation between vertices
  - an edge is a set $e = \{u, v\}$ of two vertices

Example (1)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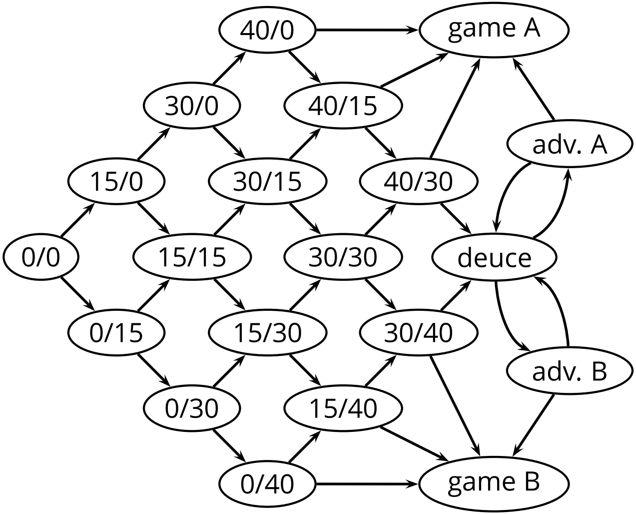
Example (2)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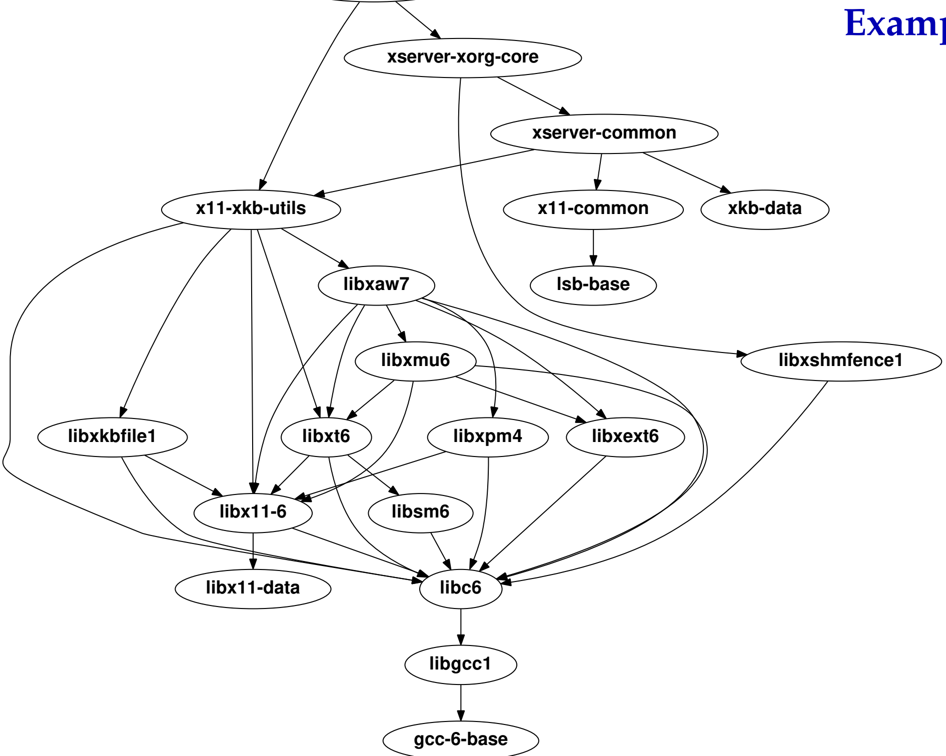
game B

Example (3)


xserver-xorg-core
xserver-common
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libxkbfile1
libxt6
libxpm4
libxext6
libx11-6
libsm6
libx11-data
libc6
libgcc1
gcc-6-base

- How do we represent a graph $G = (E, V)$ in a computer?

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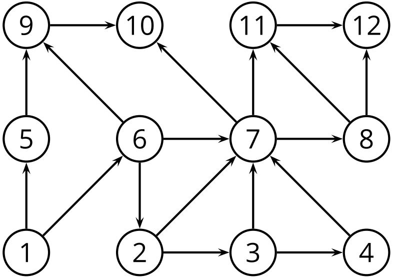
- *Adjacency-list representation*
- $V = \{1, 2, \ldots |V|\}$
- G consists of an array Adj
- A vertex $u \in V$ is represented by an element in the array Adj

Graph Representation

- How do we represent a graph $G = (E, V)$ in a computer?
- *Adjacency-list representation*
- $V = \{1, 2, \ldots |V|\}$
- G consists of an array Adj
- A vertex $u \in V$ is represented by an element in the array Adj
- $Adj[u]$ is the ***adjacency list*** of vertex u
  - the list of the vertices that are adjacent to u
  - i.e., the list of all v such that $(u, v) \in E$

Example



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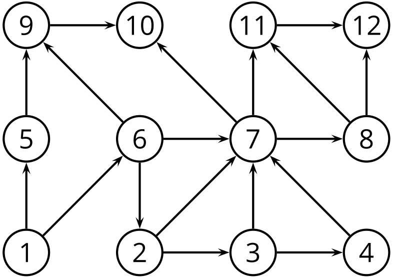
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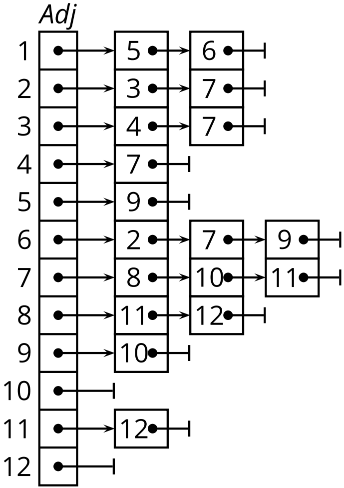
Adj
1 5 6
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7 8 10 11
8 11 12
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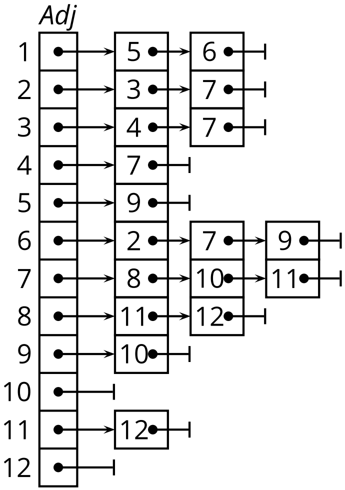
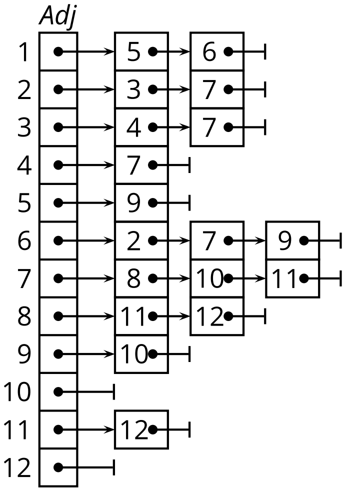
Using the Adjacency List

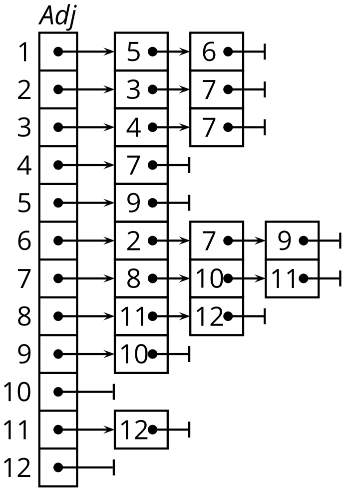


- Accessing a vertex *u*?



Using the Adjacency List

- Accessing a vertex *u*? $O(1)$
  - optimal



Using the Adjacency List

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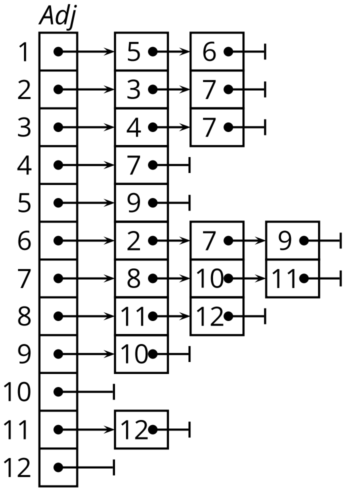
- Iteration through *V*?



Using the Adjacency List

- Accessing a vertex u? $O(1)$
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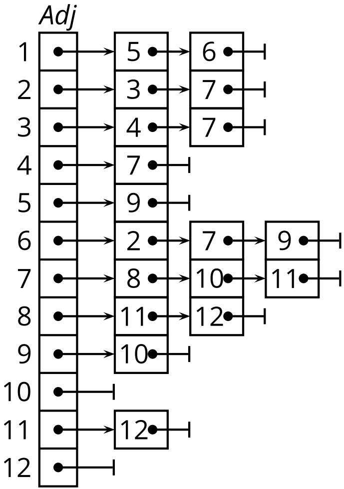
- Iteration through V? $\Theta(|V|)$
  - optimal



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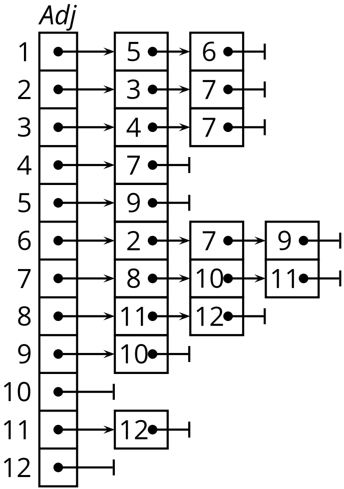
- Iteration through E?



Using the Adjacency List

- Accessing a vertex u? $O(1)$
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- Iteration through V? $\Theta(|V|)$
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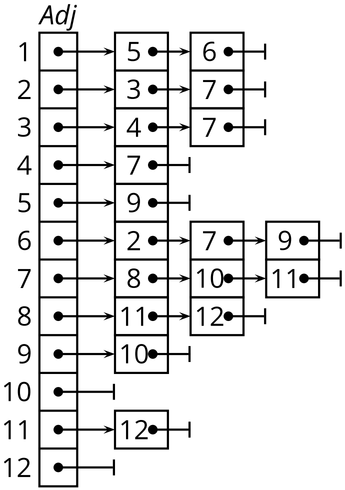
- Iteration through E? $\Theta(|V| + |E|)$
  - okay (not optimal)



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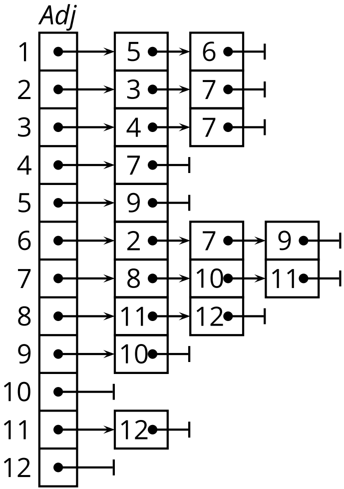

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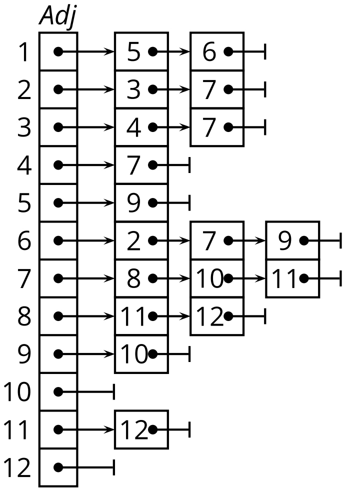

Using the Adjacency List

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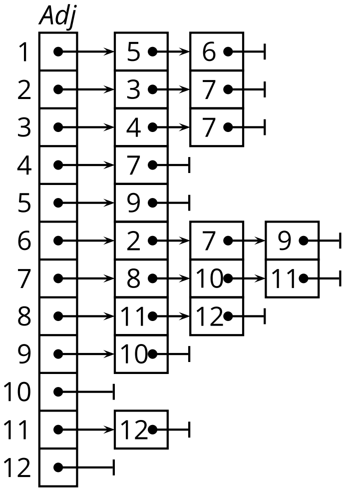
- Iteration through V? $\Theta(|V|)$
  - optimal
- Iteration through E? $\Theta(|V| + |E|)$
  - okay (not optimal)
- Checking $(u, v) \in E$? $O(|V|)$
  - bad

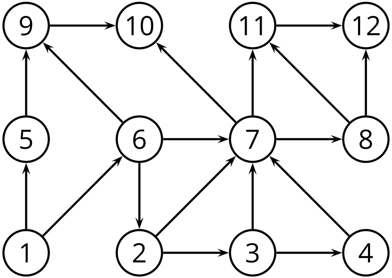


Graph Representation (2)

- *Adjacency-matrix representation*
- $V = \{1, 2, \ldots |V|\}$
- G consists of a $|V| \times |V|$ matrix A
- $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Example



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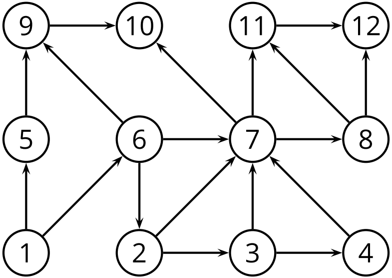

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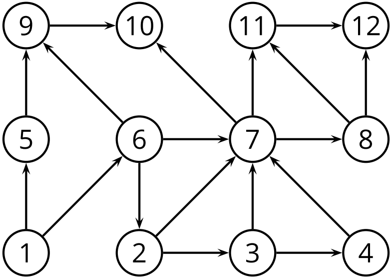
Example



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Using the Adjacency Matrix

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Using the Adjacency Matrix

- Accessing a vertex *u*?

	1	2	3	4	5	6	7	8	9	10	11	12
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Using the Adjacency Matrix

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	1	2	3	4	5	6	7	8	9	10	11	12
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Using the Adjacency Matrix

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- Iteration through *V*?

	1	2	3	4	5	6	7	8	9	10	11	12
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Using the Adjacency Matrix

- Accessing a vertex u? $O(1)$
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- Iteration through V? $\Theta(|V|)$
  - optimal

	1	2	3	4	5	6	7	8	9	10	11	12
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- Iteration through V? $\Theta(|V|)$
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- Iteration through E?

	1	2	3	4	5	6	7	8	9	10	11	12
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Using the Adjacency Matrix

- Accessing a vertex u? $O(1)$
  - optimal
- Iteration through V? $\Theta(|V|)$
  - optimal
- Iteration through E? $\Theta(|V|^2)$
  - possibly very bad

	1	2	3	4	5	6	7	8	9	10	11	12
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Using the Adjacency Matrix

- Accessing a vertex u? — $O(1)$
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- Iteration through V? — $\Theta(|V|)$
  - optimal
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  - possibly very bad
- Checking $(u, v) \in E$? — $O(1)$
  - optimal

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Space Complexity

- Adjacency-list representation

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$$\boxed{\Theta(|V| + |E|)}$$

Space Complexity

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optimal

Space Complexity

- Adjacency-list representation

$$\boxed{\Theta(|V| + |E|)}$$

  optimal

- Adjacency-matrix representation

- Adjacency-list representation

$$\boxed{\Theta(|V| + |E|)}$$

optimal

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$$\boxed{\Theta(|V|^2)}$$

Space Complexity

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- Adjacency-matrix representation

$$\boxed{\Theta(|V|^2)}$$

possibly very bad

- Adjacency-list representation

$$\Theta(|V| + |E|)$$

  optimal

- Adjacency-matrix representation

$$\Theta(|V|^2)$$

  possibly very bad

- When is the adjacency-matrix "very bad"?

Choosing a Graph Representation

- Adjacency-list representation
  - generally good, especially for its optimal space complexity
  - bad for ***dense*** graphs and algorithms that require random access to edges
  - preferable for ***sparse*** graphs or graphs with ***low degree***

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- Adjacency-list representation
  - generally good, especially for its optimal space complexity
  - bad for ***dense*** graphs and algorithms that require random access to edges
  - preferable for ***sparse*** graphs or graphs with ***low degree***

- Adjacency-matrix representation
  - suffers from a bad space complexity
  - good for algorithms that require random access to edges
  - preferable for ***dense*** graphs

Breadth-First Search

- One of the simplest but also a fundamental algorithm

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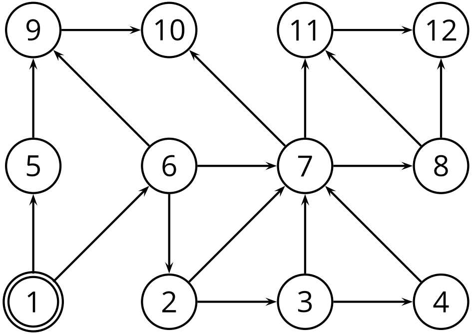
- *Input:* $G = (V, E)$ and a vertex $s \in V$
  - explores the graph, touching all vertices that are reachable from s
  - iterates through the vertices at increasing distance (edge distance)
  - computes the distance of each vertex from s
  - produces a ***breadth-first tree*** rooted at s
  - works on both *directed* and *undirected* graphs

Example



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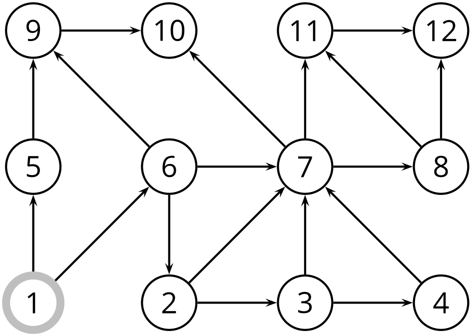

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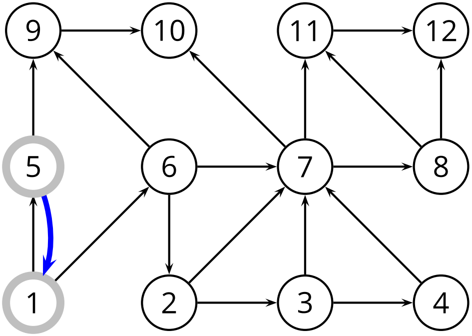

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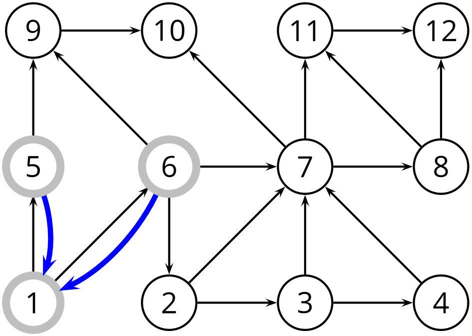

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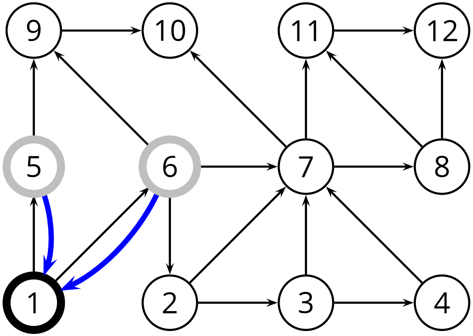

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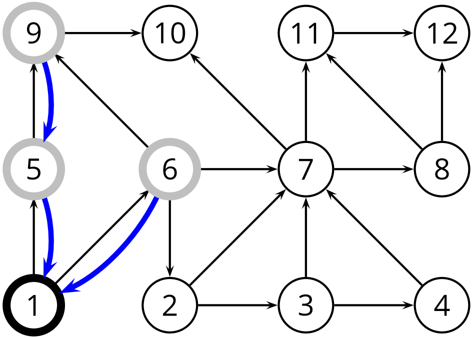

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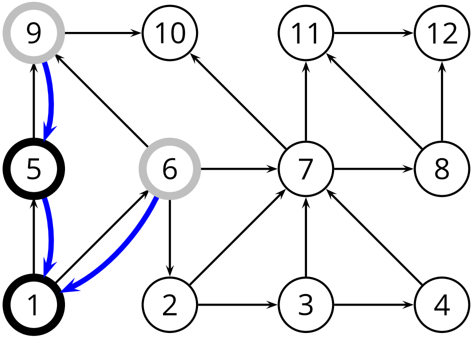

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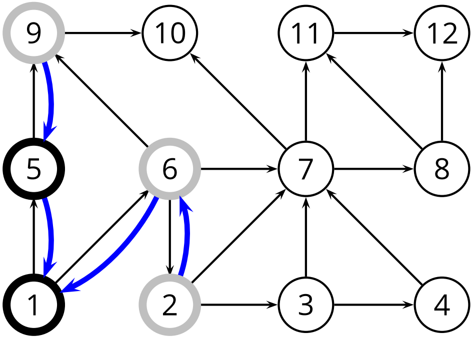

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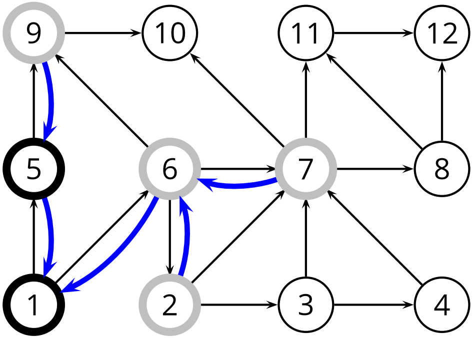

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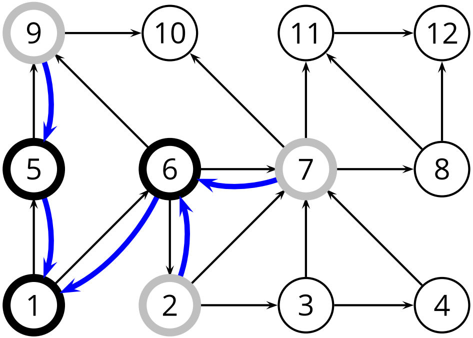

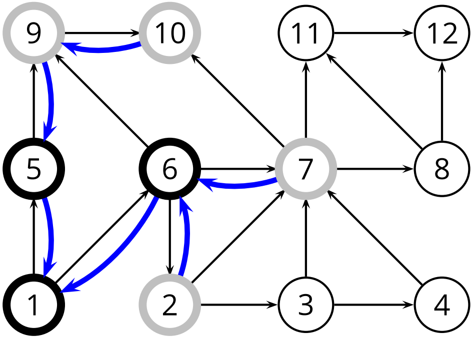
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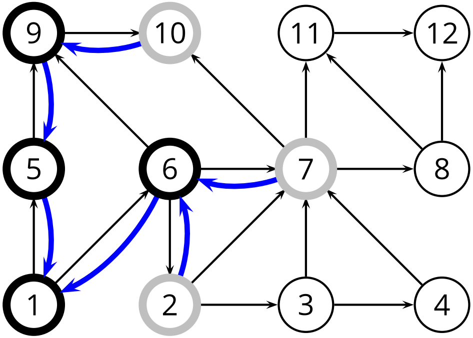
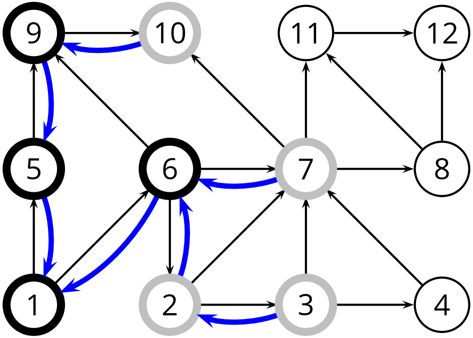
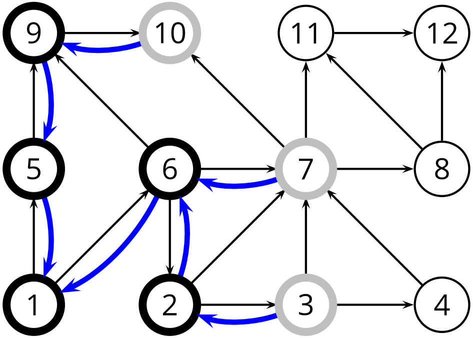


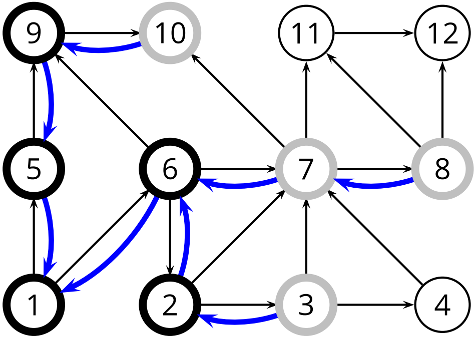
Example

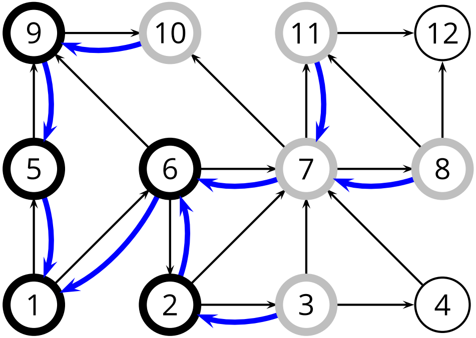


Example



Example




9
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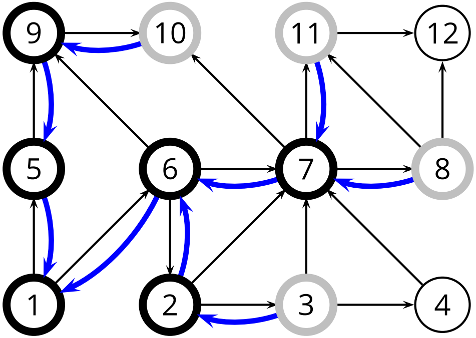
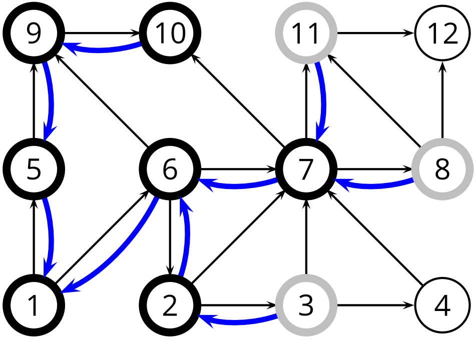
Example




9
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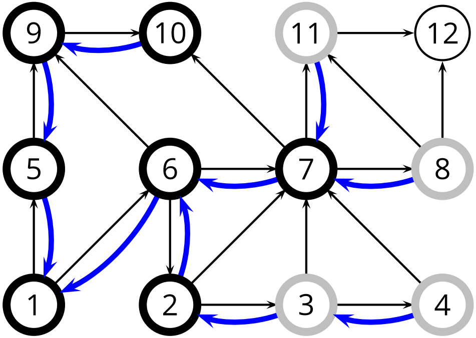
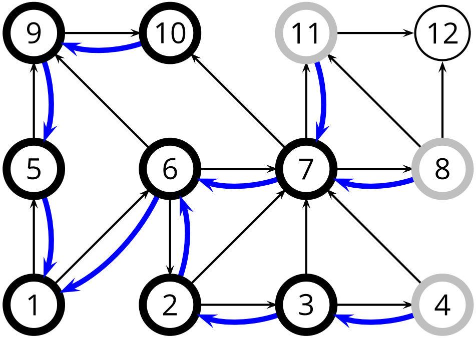
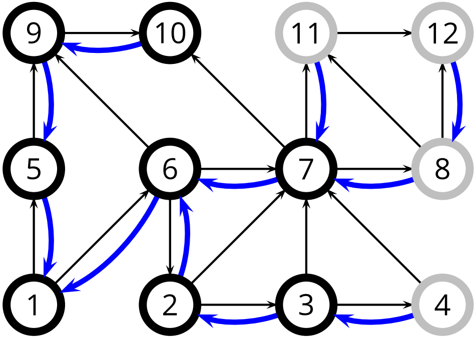
Example



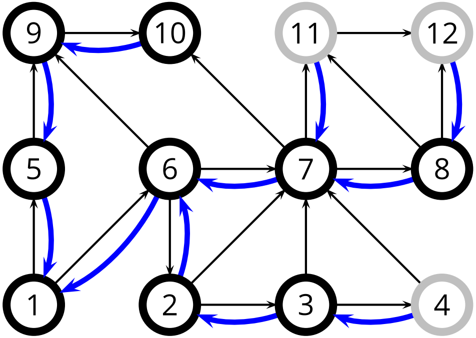
Example



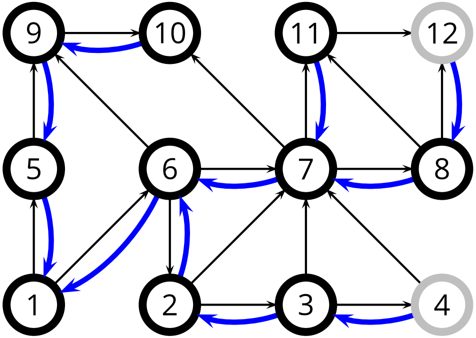
Example



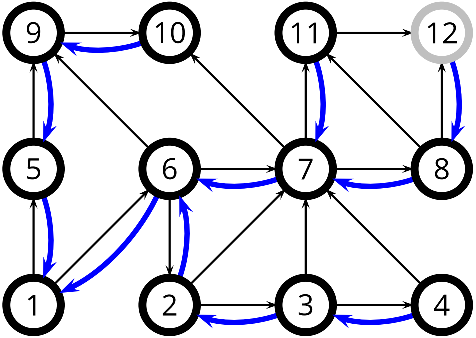
Example



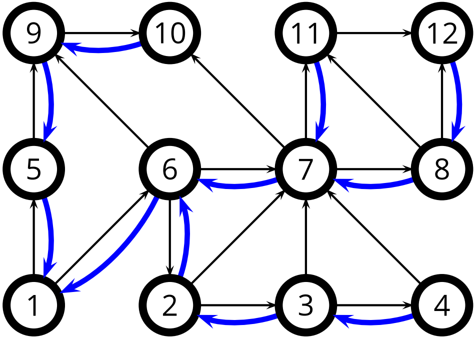
Example



Example

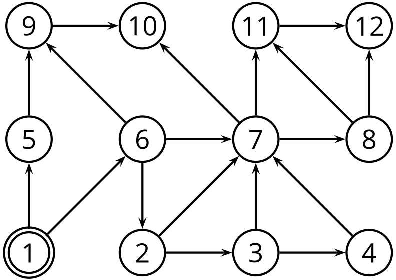


Example



BFS Algorithm

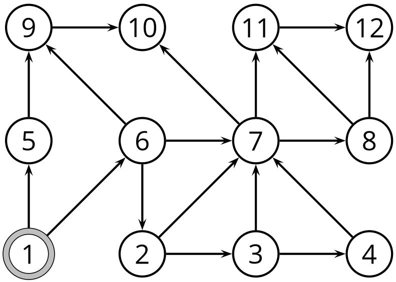
```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```



BFS Algorithm

BFS(G, s)

```
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

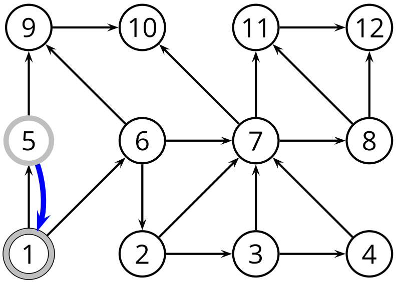


$u = 1$

$Q = \emptyset$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

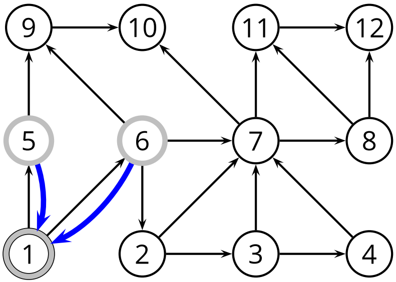


$u = 1$

$Q = \{5\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

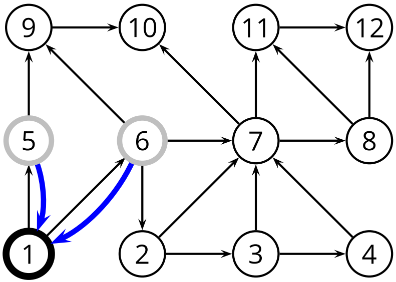


$u = 1$

$Q = \{5, 6\}$

BFS Algorithm

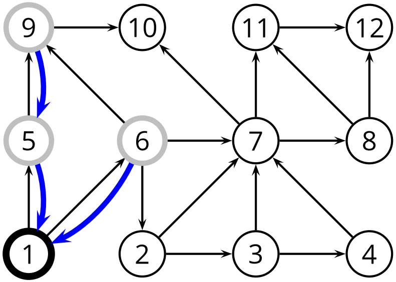
```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```



$u = 5$

$Q = \{6\}$

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

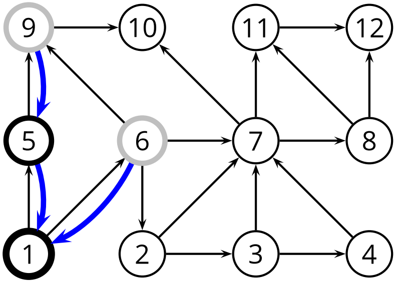


$u = 5$

$Q = \{6, 9\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

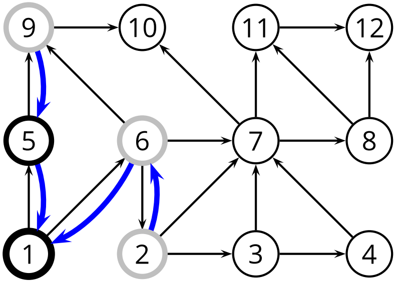


$u = 6$

$Q = \{9\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

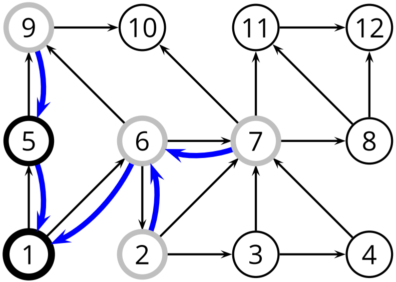


$u = 6$

$Q = \{9, 2, 7\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

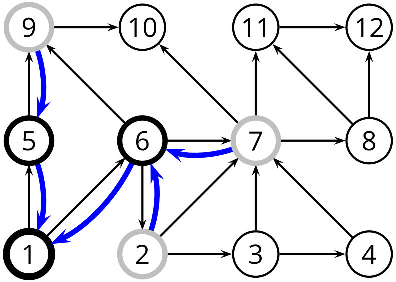


$u = 6$

$Q = \{9, 2, 7\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

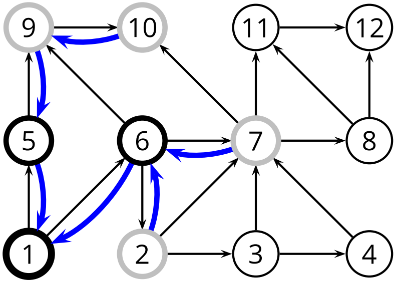


$u = 9$

$Q = \{2, 7\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

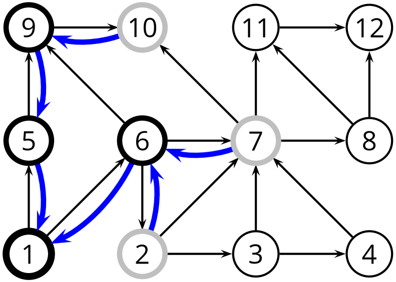


$u = 9$

$Q = \{2, 7, 10\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

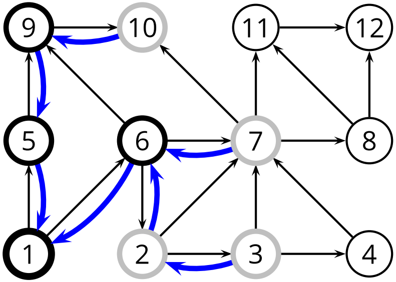


$u = 2$

$Q = \{7, 10\}$

BFS Algorithm

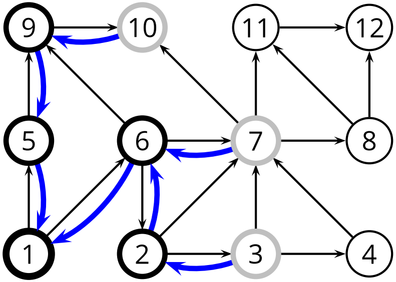
```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```



$u = 2$

$Q = \{7, 10, 3\}$

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

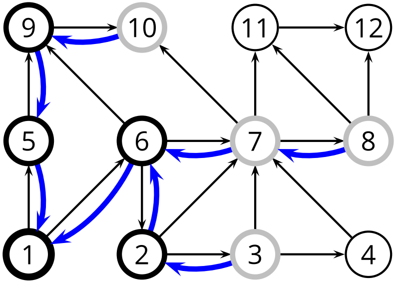


$u = 7$

$Q = \{10, 3\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

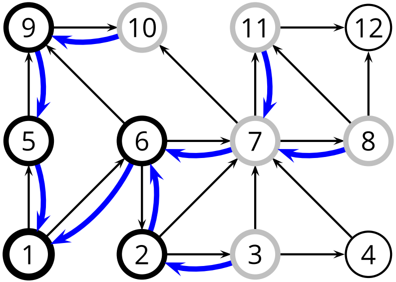


$u = 7$

$Q = \{10, 3, 8\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

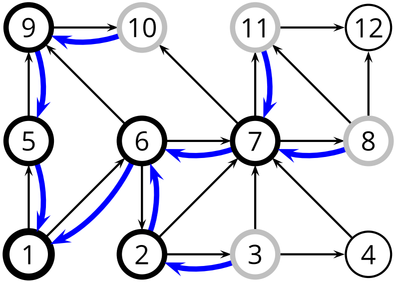


$u = 7$

$Q = \{10, 3, 8, 11\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

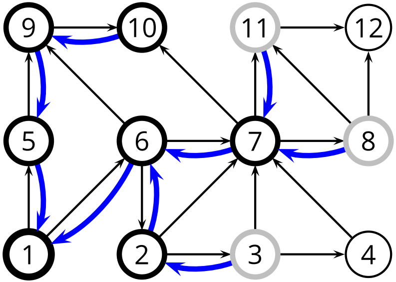


$u = 10$

$Q = \{3, 8, 11\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

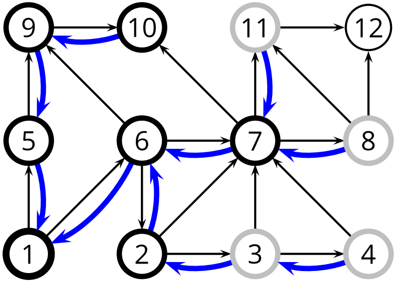


$u = 3$

$Q = \{8, 11\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

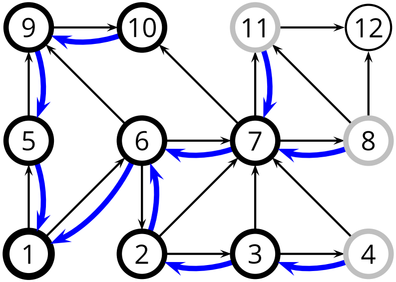


$u = 3$

$Q = \{8, 11, 4\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```



$u = 8$

$Q = \{11, 4\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
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```

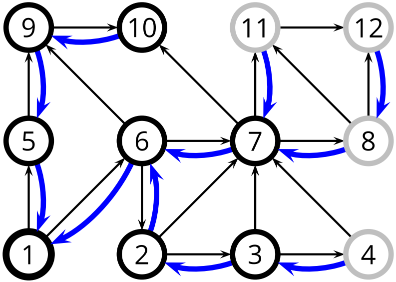

$u = 8$

$Q = \{11, 4, 12\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
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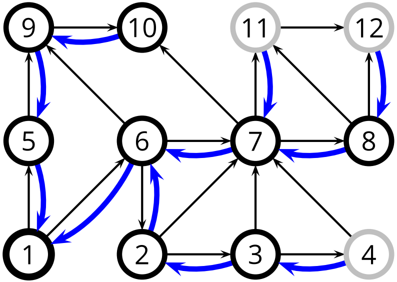

$u = 11$

$Q = \{4, 12\}$

BFS Algorithm

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for each vertex u ∈ V(G) \ {s}
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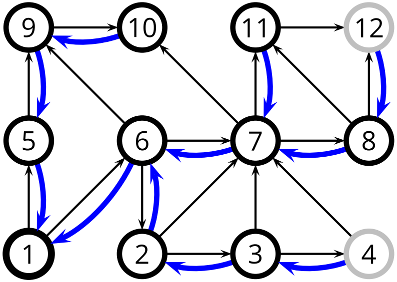

$u = 4$

$Q = \{12\}$

```
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for each vertex u ∈ V(G) \ {s}
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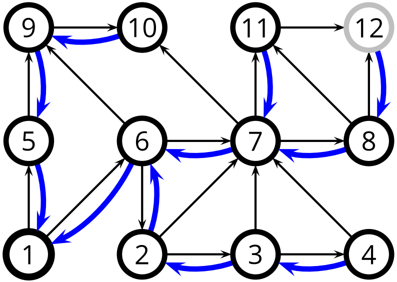

$u = 12$

$Q = \emptyset$

BFS Algorithm

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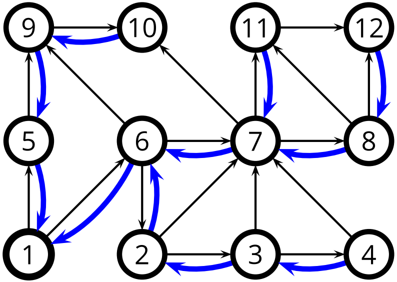

Complexity of BFS

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- So, $O(|V| + |E|)$

Depth-First Search

- Immediately follow the links of the most recently-visited vertex, then backtrack when you reach a dead-end
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  - explores the graph, touching *all vertices*
  - produces a ***depth-first forest***, consisting of all the ***depth-first trees*** defined by the DFS exploration
  - associates ***two time-stamps*** to each vertex
    - $d[u]$ records when u is first discovered
    - $f[u]$ records when DFS finishes examining u's edges, and therefore backtracks from u

DFS Algorithm

```
DFS(G)
for each vertex u ∈ V(G)
        color[u] = WHITE
        π[u] = NIL
time = 0 // "global" variable
for each vertex u ∈ V(G)
        if color[u] == WHITE
                DFS-VISIT(u)
```

```
DFS-VISIT(u)
color[u] = GREY
time = time + 1
d[u] = time
for each v ∈ Adj[u]
        if color[v] == WHITE
                π[v] = u
                DFS-VISIT(v)
color[u] = BLACK
time = time + 1
f[u] = time
```

Complexity of DFS

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- So, the overall complexity is $\Theta(|V| + |E|)$

Applications of DFS: Topological Sort

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- Given a *directed acyclic graph* (DAG)
  - find an ordering of vertices such that you only end up with *forward links*

- Example: dependencies in software packages
  - find an installation order for a set of software packages
  - such that every package is installed only after all the packages it depends on

Topological Sort Algorithm

Topological Sort Algorithm

```
TOPOLOGICAL-SORT(G)
1   DFS(G)
2   output V sorted in reverse order of f[·]
```

TOPOLOGICAL-SORT(G)

1 **DFS**(G)
2 output V sorted in reverse order of $f[\cdot]$