Elementary Data Structures and Hash Tables

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Outline

- Common concepts and notation
- Stacks
- Queues
- Linked lists
- Trees
- Direct-access tables
- Hash tables

- A ***data structure*** is a way to organize and store information
  - to facilitate access, or for other purposes

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- A data structure stores ***data*** and possibly ***meta-data***
  - e.g., a *heap* needs an array A to store the keys, plus a variable $A.\mathit{heap\text{-}size}$ to remember how many elements are in the heap

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- *Implementation*
  - using an array
  - using a linked list
  - ...

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if S.top == 0
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STACK-EMPTY(S)

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if S.top == 0
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PUSH(S,X)

```
S.top = S.top + 1
S[S.top] = x
```

POP(S)

```
if STACK-EMPTY(S)
        error "underflow"
else S.top = S.top − 1
        return S[S.top + 1]
```

Queue

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- *Implementation*
  - Q is an array of fixed length $Q.length$
    - i.e., Q holds at most $Q.length$ elements
    - enqueueing more than Q elements causes an “overflow” error
  - $Q.head$ is the position of the “head” of the queue
  - $Q.tail$ is the first empty position at the tail of the queue

Enqueue

ENQUEUE(Q,X)

```
if Q.queue-full
        error "overflow"
else Q[Q.tail] = x
        if Q.tail < Q.length
                Q.tail = Q.tail + 1
        else Q.tail = 1
        if Q.tail == Q.head
                Q.queue-full = TRUE
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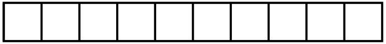

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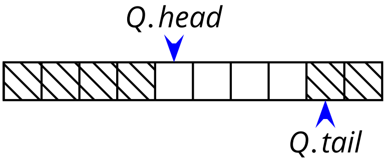

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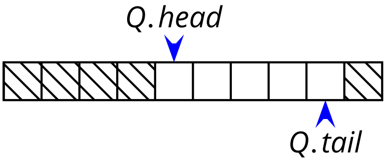

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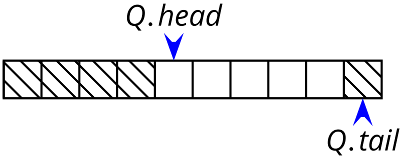

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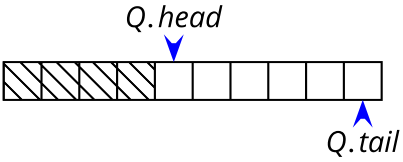

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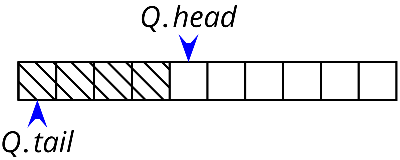

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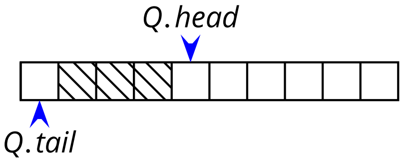

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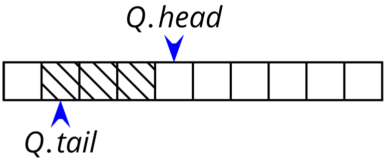

Dequeue

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DEQUEUE(Q)
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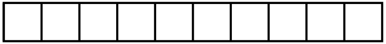

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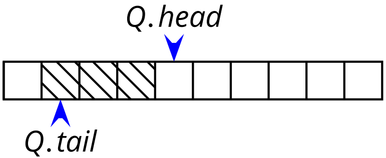

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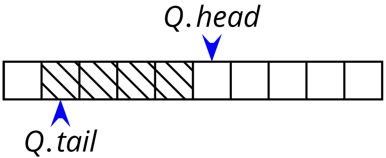

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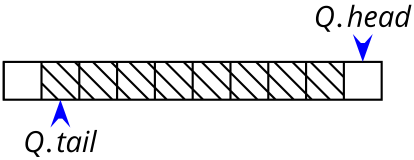

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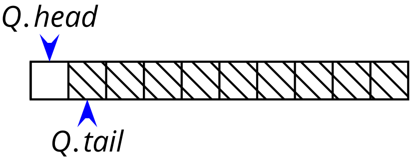

- *Interface*
  - **LIST-INSERT**(L, x) adds element x at beginning of a list L
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  - **LIST-SEARCH**(L, k) finds an element whose key is k in a list L

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- *Implementation*
  - a *doubly-linked* list
  - each element x has two “links” $x.prev$ and $x.next$ to the previous and next elements, respectively
  - each element x holds a key $x.key$
  - it is convenient to have a dummy “sentinel” element $L.nil$

Linked List With a “Sentinel”

List-Init(L)

```
L.nil.prev = L.nil
L.nil.next = L.nil
```

List-Insert(L, x)

```
x.next = L.nil.next
L.nil.next.prev = x
L.nil.next = x
x.prev = L.nil
```

List-Search(L, k)

```
x = L.nil.next
while x ≠ L.nil ∧ x.key ≠ k
        x = x.next
return x
```

- *Structure*
  - fixed branching
  - unbounded branching

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- *Implementation*
  - for each node $x \neq T.root$, $x.parent$ is x's parent node
  - fixed branching:
    e.g., $x.left\text{-}child$ and $x.right\text{-}child$ in a *binary tree*
  - unbounded branching:
    $x.left\text{-}child$ is x's first (leftmost) child
    $x.right\text{-}sibling$ is x closest sibling to the right

Algorithm	*Complexity*

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STACK-EMPTY	

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STACK-EMPTY	$O(1)$
PUSH	

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PUSH	$O(1)$
POP	$O(1)$
ENQUEUE	$O(1)$
DEQUEUE	$O(1)$
LIST-INSERT	

Algorithm	*Complexity*
Stack-Empty	$O(1)$
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List-Search	$\Theta(n)$

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  - ***hash tables***

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```
DIRECT-ADDRESS-INSERT(T, k)
1  T[k] = TRUE
```

```
DIRECT-ADDRESS-DELETE(T, k)
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```
DIRECT-ADDRESS-SEARCH(T, k)
1  return T[k]
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- The ***space complexity*** is $\Theta(|U|)$
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- *Can we have the benefits of a direct-address table but with a table of reasonable size?*

- *Idea*
  - use a table T with $|T| \ll |U|$
  - map each key $k \in U$ to a position in T, using a ***hash function***

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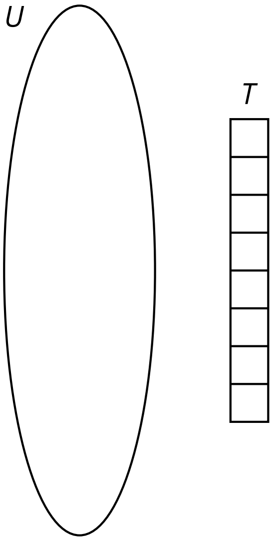
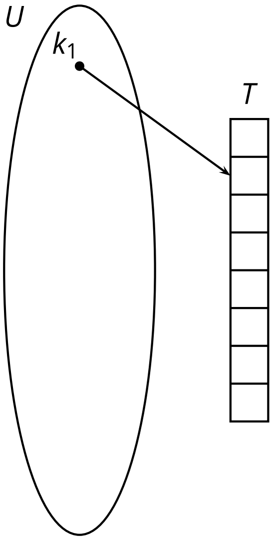
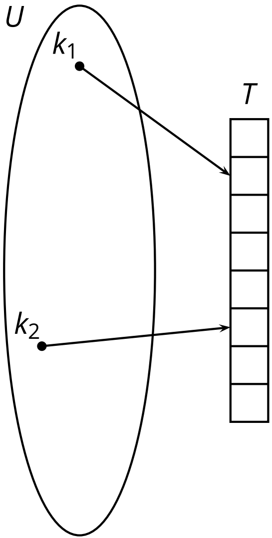
What if two distinct keys $k_1 \neq k_2$ *collide?* (I.e., $h(k_1) = h(k_2)$)

Hash Table

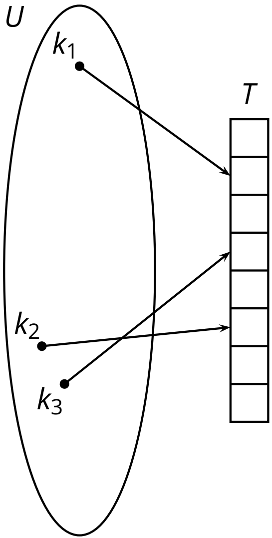


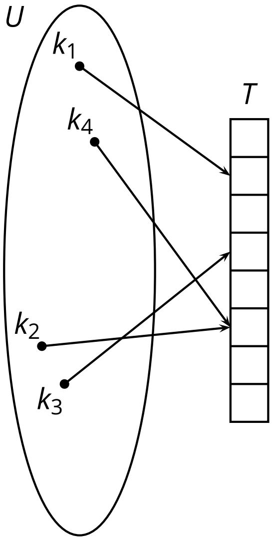

U
k_1
T



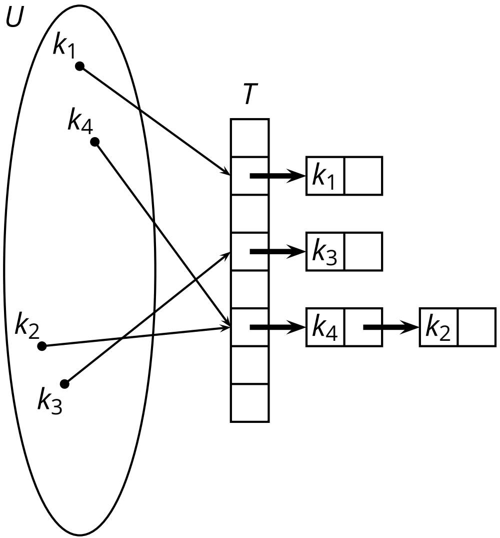
U
k_1
T
k_2

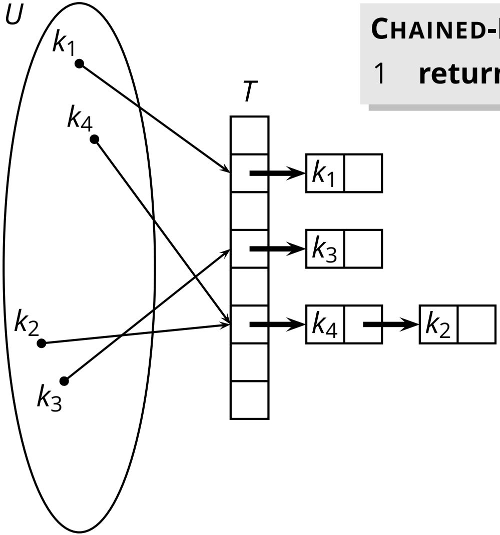
Hash Table




U
k_1
k_4
k_2
k_3
T

Hash Table





CHAINED-HASH-INSERT(T, k)

1 **return** LIST-INSERT$(T[h(k)], k)$

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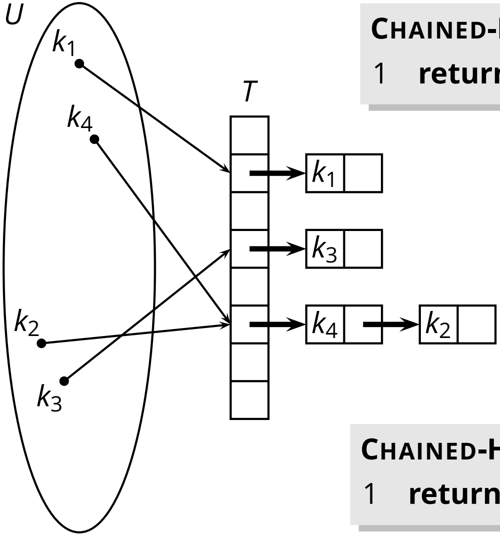

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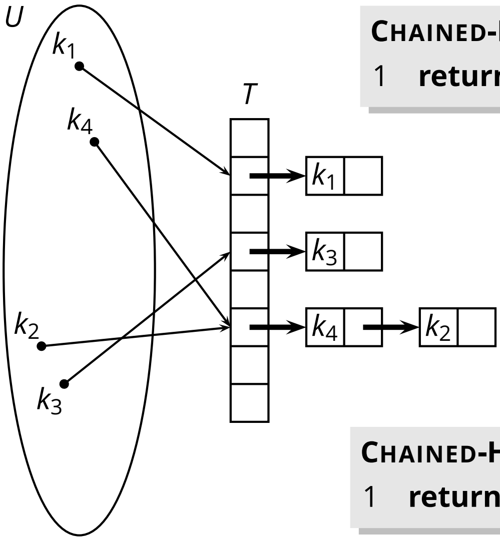


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load factor

$$\alpha = \frac{n}{|T|}$$

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CHAINED-HASH-SEARCH(T, k)
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(The formalism is actually a bit more complicated.)

Analysis

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(The formalism is actually a bit more complicated.)

- So, given n distinct keys, the expected length n_i of the linked list at position i is

$$E[n_i] = \frac{n}{|T|} = \alpha$$

- We assume ***uniform hashing*** for our hash function $h : U \to \{1 \dots |T|\}$ (where $|T| = T.length$)

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(The formalism is actually a bit more complicated.)

- So, given n distinct keys, the expected length n_i of the linked list at position i is

$$E[n_i] = \frac{n}{|T|} = \alpha$$

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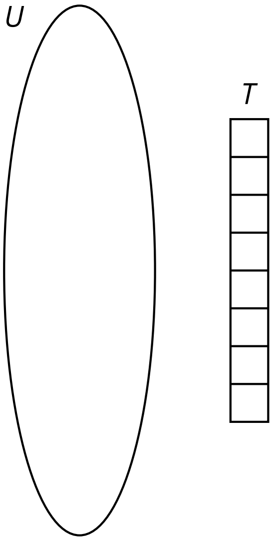
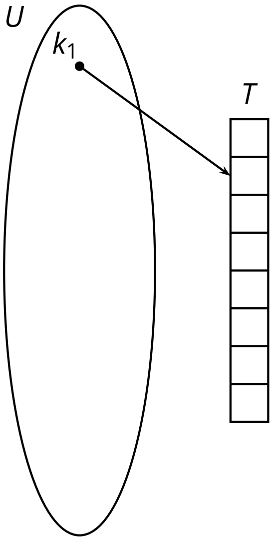
- Therefore, the complexity of **CHAINED-HASH-SEARCH** is

$$\boxed{\Theta(1 + \alpha)}$$

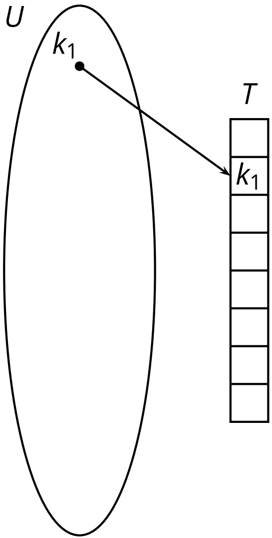
Open-Address Hash Table



Open-Address Hash Table



Open-Address Hash Table



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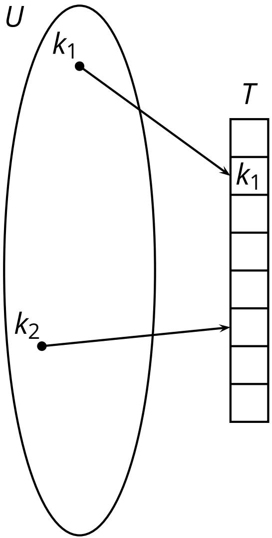

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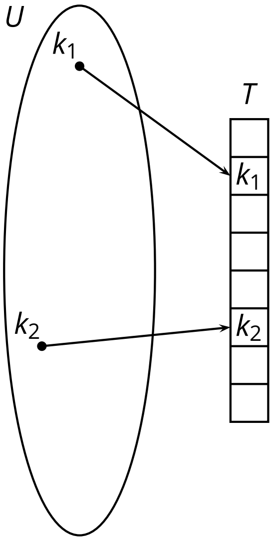

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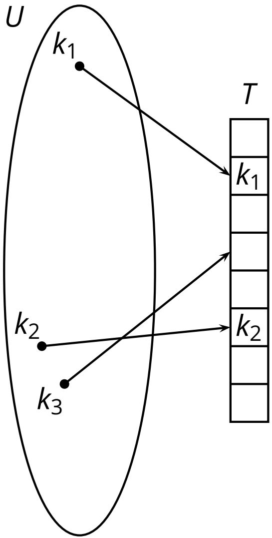
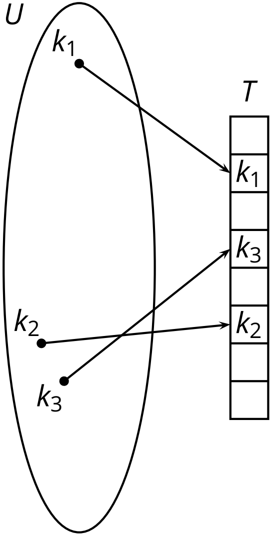
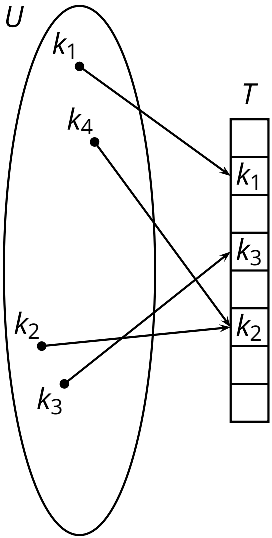


Open-Address Hash Table



Open-Address Hash Table



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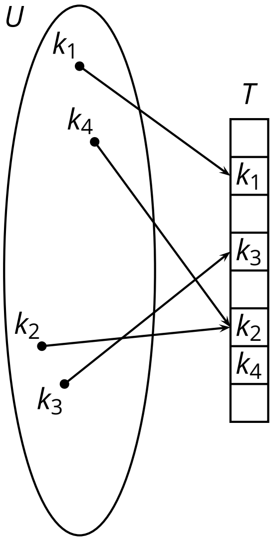

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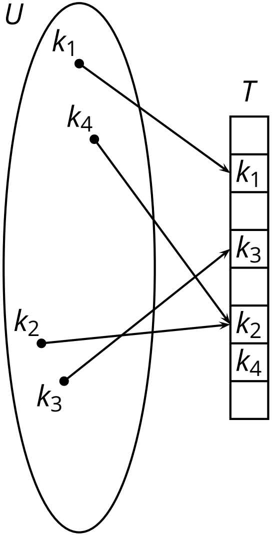


```
HASH-INSERT(T, k)
  j = h(k)
  for i = 1 to T.length
      if T[j] == NIL
          T[j] = k
          return j
      elseif j < T.length
          j = j + 1
      else j = 1
  error "overflow"
```

Open-Addressing (2)

- *Idea:* instead of using linked lists, we can store all the elements in the table
  - this implies $\alpha \leq 1$

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- *Idea:* instead of using linked lists, we can store all the elements in the table
  - this implies $\alpha \leq 1$
- When a collision occurs, we simply find another free cell in T
- A sequential “probe” may not be optimal
  - can you figure out why?

Open-Addressing (3)

```
HASH-INSERT(T, k)
for i = 1 to T.length
    j = h(k, i)
    if T[j] == NIL
        T[j] = k
        return j
error "overflow"
```

```
HASH-INSERT(T, k)
for i = 1 to T.length
    j = h(k, i)
    if T[j] == NIL
        T[j] = k
        return j
error "overflow"
```

- Notice that $h(k,\cdot)$ must be a ***permutation***
  - i.e., $h(k,1), h(k,2), \ldots, h(k,|T|)$ must cover the entire table T