# Binary Search Trees

Antonio Carzaniga

Faculty of Informatics
Università della Svizzera italiana

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# Outline

- Binary search trees
- Randomized binary search trees

- A ***binary search tree*** implements of a *dynamic set*
  - over a ***totally ordered domain***

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  - *iteration:* **TREE-SUCCESSOR**$(x)$ and **TREE-PREDECESSOR**$(x)$ find the successor and predecessor, respectively, of an element $x$

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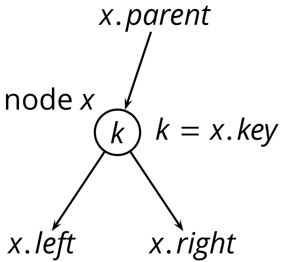
Node $x$

- $x.parent$ is the parent of node $x$
- $x.key$ is the key stored in node $x$
- $x.left$ is the left child of node $x$
- $x.right$ is the right child of node $x$



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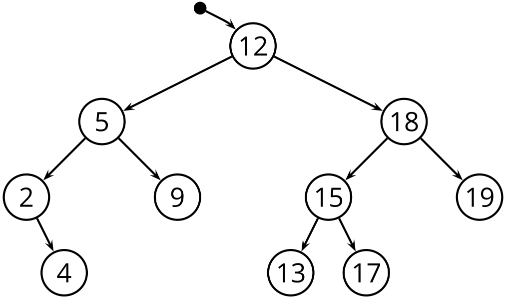

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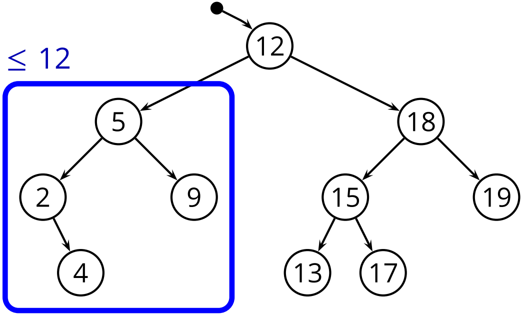

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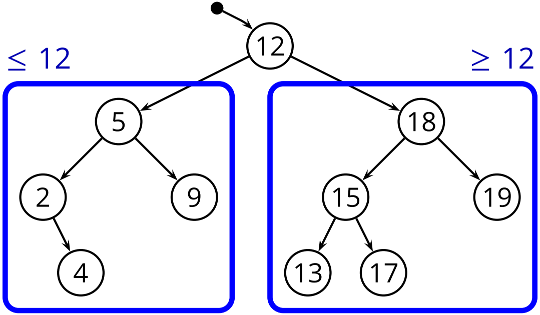

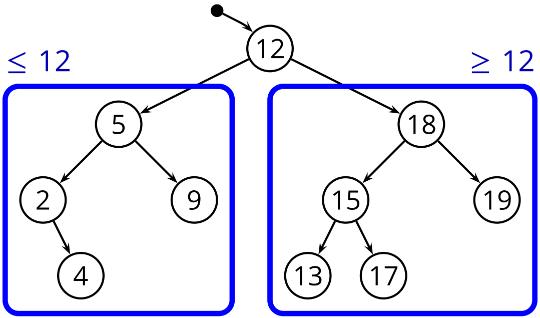


- ***Binary-search-tree property***
  - for all nodes $x$, $y$, and $z$
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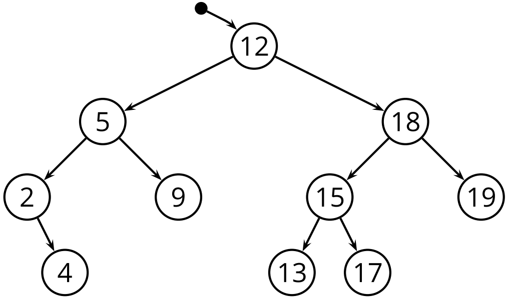

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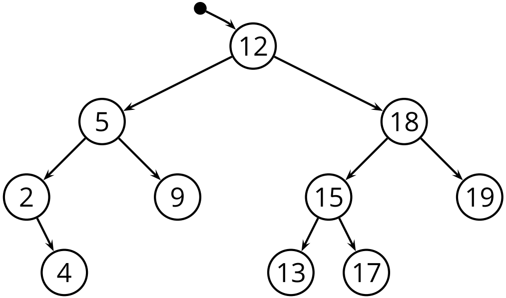


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And then we need a “starter” procedure

```
INORDER-TREE-WALK-START(T)
INORDER-TREE-WALK(T.root)
```

# Pre-Order Tree Walk

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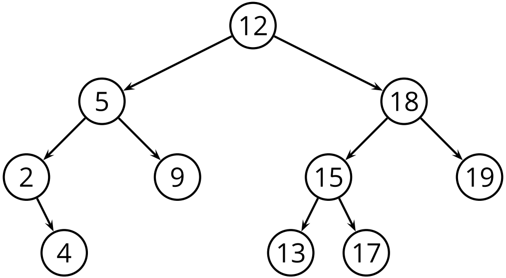

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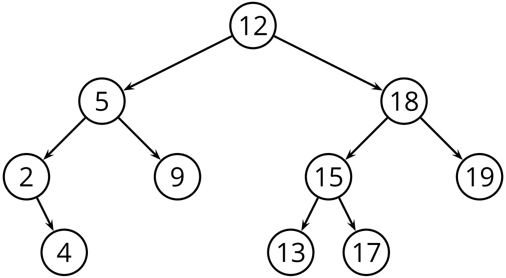


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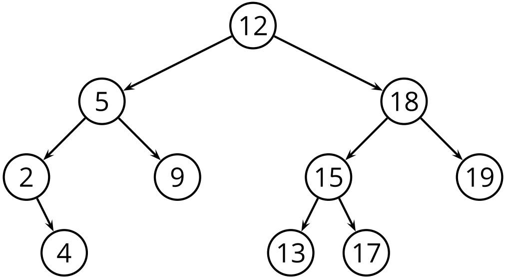

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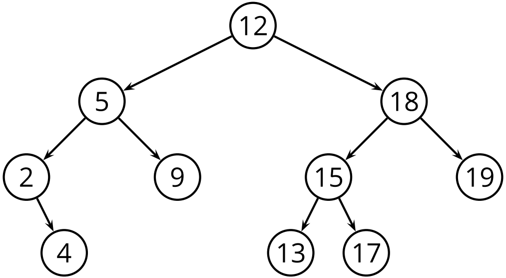


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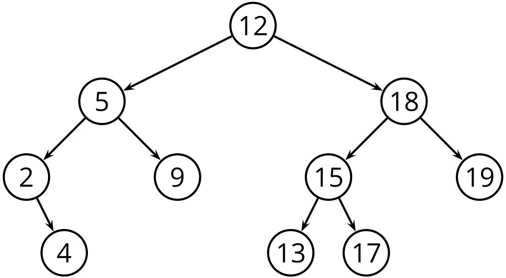

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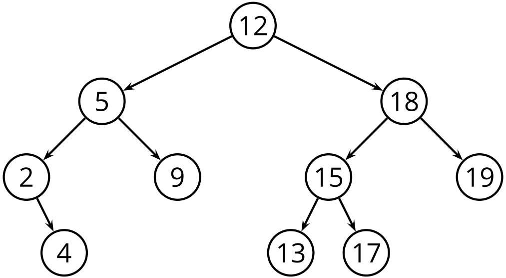


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- Can we do better? No!
  - the length of the output is $\Theta(n)$

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- So, the minimum key is in all the way to the left
  - similarly, the maximum key is all the way to the right

```
TREE-MINIMUM(x)
while x.left ≠ NIL
    x = x.left
return x
```

```
TREE-MAXIMUM(x)
while x.right ≠ NIL
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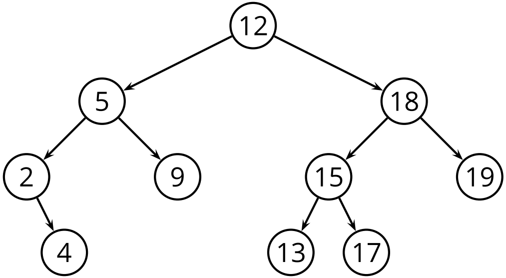

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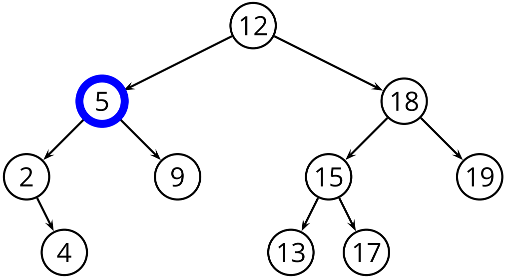

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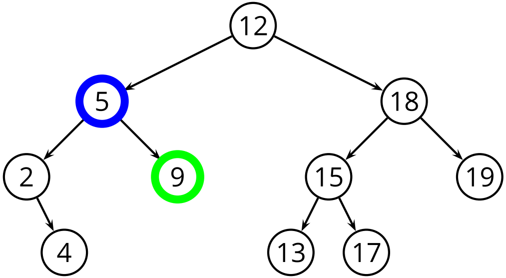

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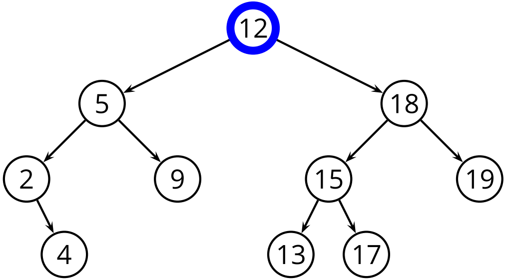

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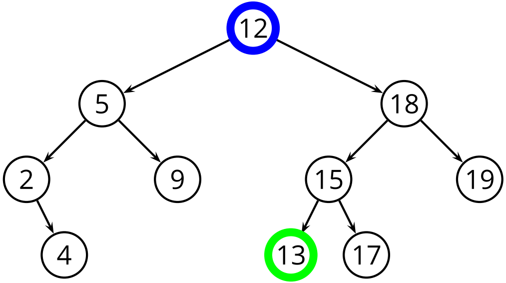

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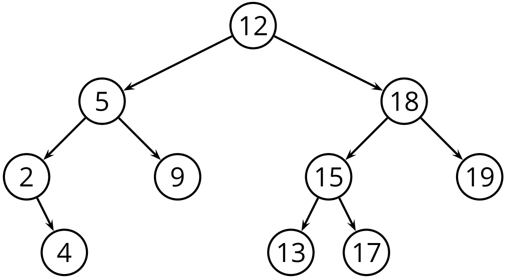


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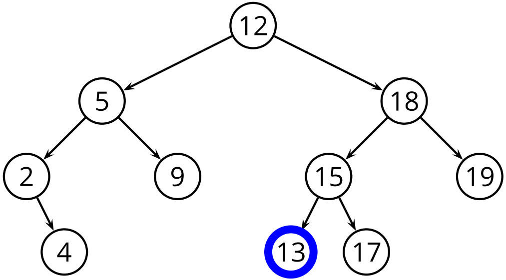


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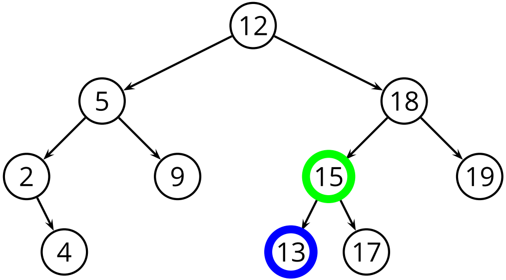


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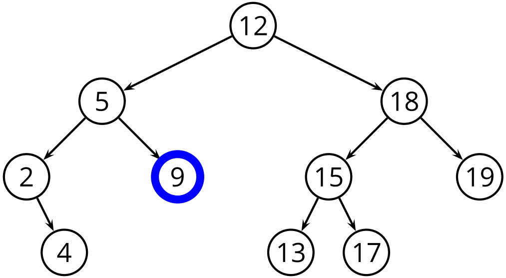


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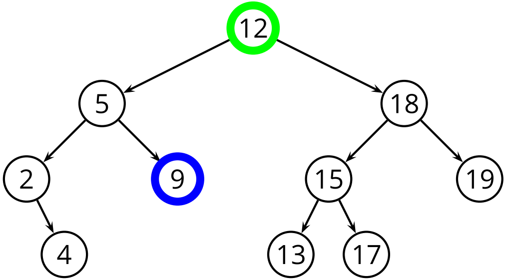


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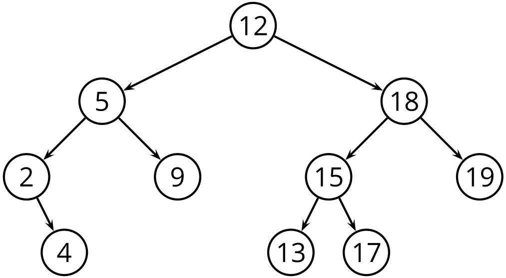


- The successor of *x* is the *minimum* of the *right* subtree of *x*, if that exists
- Otherwise it is the *first ancestor a* of *x* such that *x* falls in the *left* subtree of *a*

# Successor and Predecessor(2)

```
TREE-SUCCESSOR(x)
if x.right ≠ NIL
        return TREE-MINIMUM(x.right)
y = x.parent
while y ≠ NIL and x = y.right
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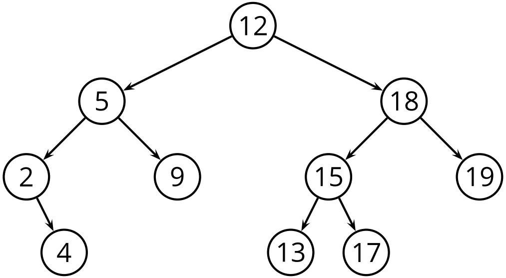

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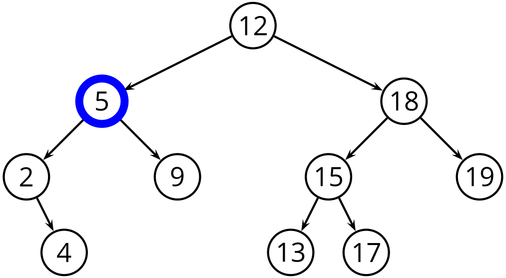

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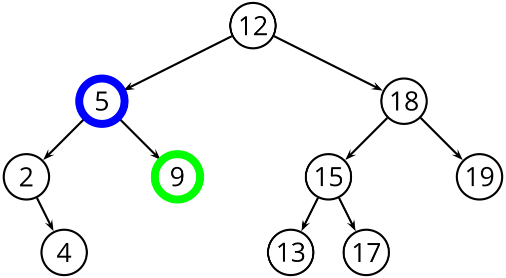

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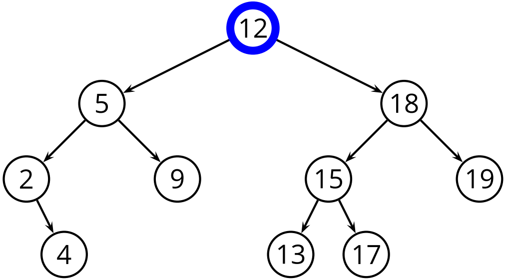

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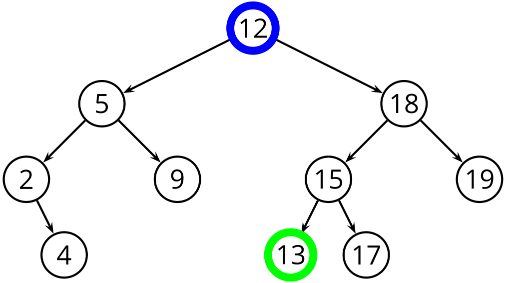

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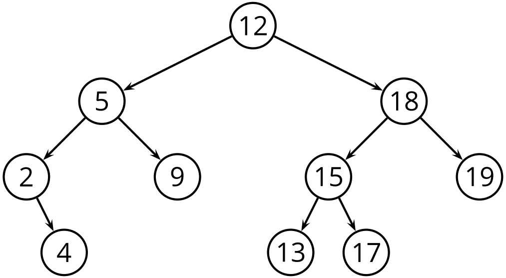

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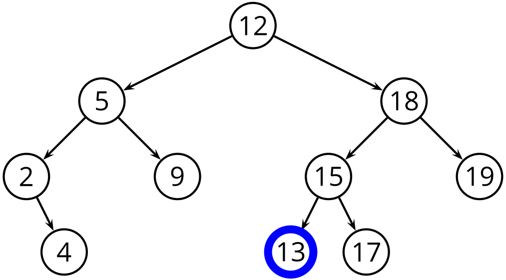

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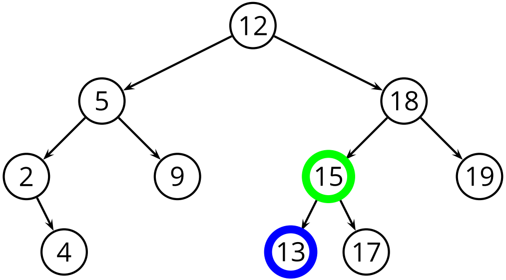

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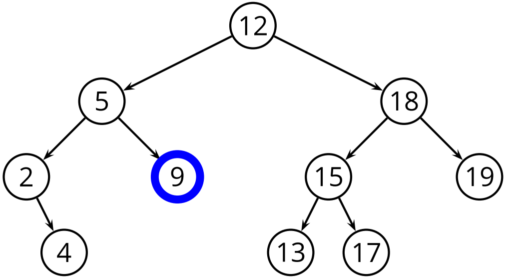

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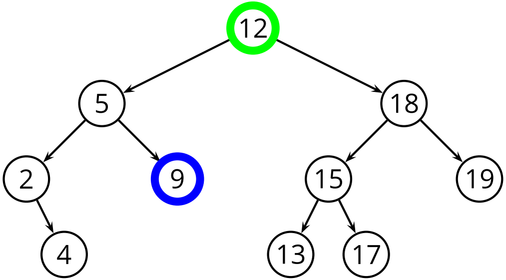

- *Binary search* (thus the name of the tree)

# Search

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```
TREE-SEARCH(x, k)
if x == NIL or k == x.key
        return x
if k < x.key
        return TREE-SEARCH(x.left, k)
else return TREE-SEARCH(x.right, k)
```

- *Binary search* (thus the name of the tree)

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- Is this correct?

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$$\boxed{T(n) = O(n)}$$

# Search (2)

- Iterative *binary search*

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```
ITERATIVE-TREE-SEARCH(T, k)
x = T.root
while x ≠ NIL ∧ k ≠ x.key
    if k < x.key
        x = x.left
    else x = x.right
return x
```

# Insertion

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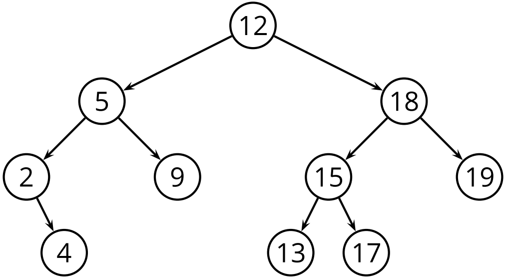

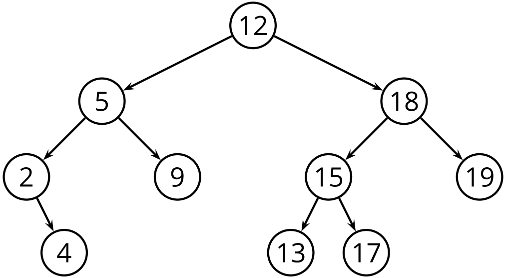


- *Idea*
  - in order to insert $x$, we *search* for $x$ (more precisely $x.key$)
  - if we don't find it, we add it where the search stopped

# Insertion (2)

```
TREE-INSERT(T, z)
y = NIL
x = T.root
while x ≠ NIL
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.parent = y
if y = NIL
    T.root = z
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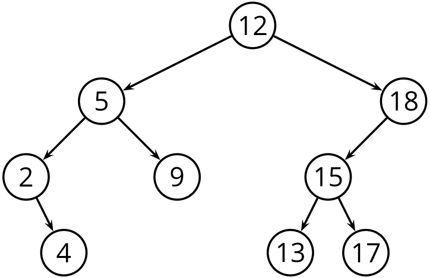

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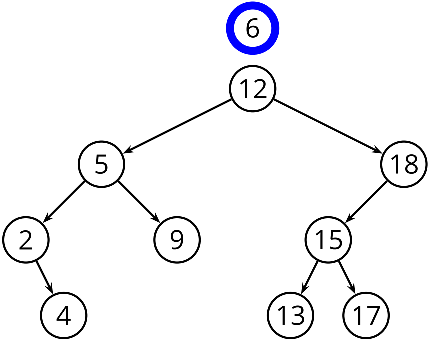

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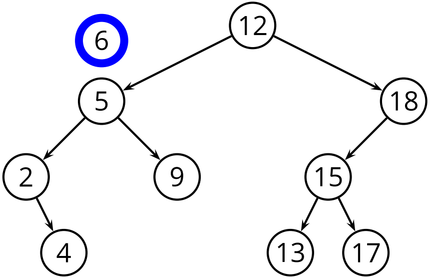

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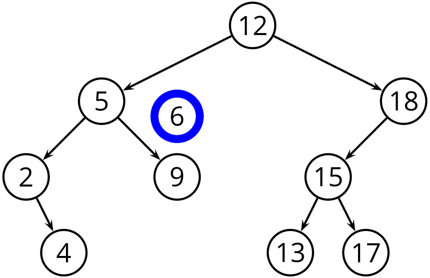

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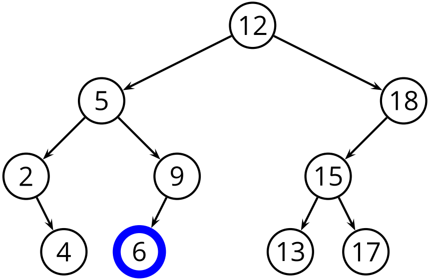

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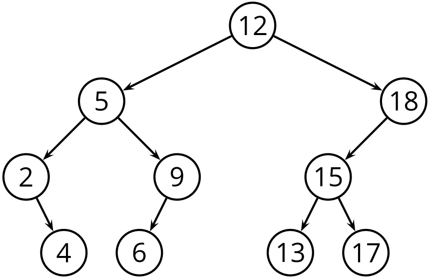

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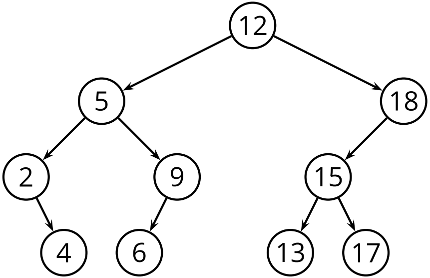


$$\boxed{T(n) = \Theta(h)}$$

# Observation

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- *Idea:* use randomization to turn all cases in the average case

# Randomized Insertion

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- *Idea 2:* we can obtain a random permutation of the input sequence by randomly alternating two insertion procedures
  - *tail insertion:* this is what **TREE-INSERT** does
  - *head insertion:* for this we need a new procedure **TREE-ROOT-INSERT**
    - inserts $n$ in $T$ as if $n$ was inserted as the first element

# Randomized Insertion (2)

```
TREE-RANDOMIZED-INSERT1(T, z)
r = uniformly random value from {1, ..., t.size + 1}
if r = 1
        TREE-ROOT-INSERT(T, z)
else TREE-INSERT(T, z)
```

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  - no, clearly the last element can only go to the top or to the bottom
- It is true that any node has the same probability of being inserted at the top
  - this suggests a recursive application of this same procedure

# Randomized Insertion (3)

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```
TREE-RANDOMIZED-INSERT(t, z)
if t = NIL
    return z
r = uniformly random value from {1, . . . , t.size + 1}
if r = 1                                  // Pr[r = 1] = 1/(t.size + 1)
    z.size = t.size + 1
    return TREE-ROOT-INSERT(t, z)
if z.key < t.key
    t.left = TREE-RANDOMIZED-INSERT(t.left, z)
else t.right = TREE-RANDOMIZED-INSERT(t.right, z)
t.size = t.size + 1
return t
```

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t.size = t.size + 1
return t
```

- Looks like this one really simulates a random permutation…

# Rotation

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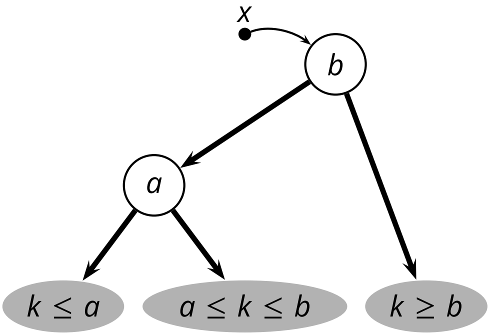

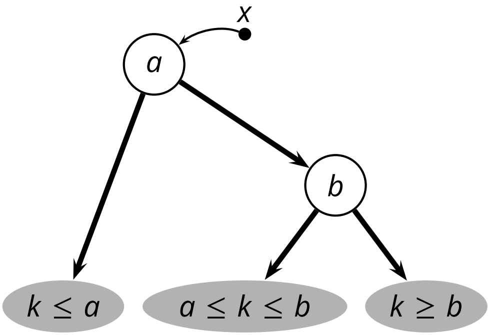


- $x = \textsc{Right-Rotate}(x)$

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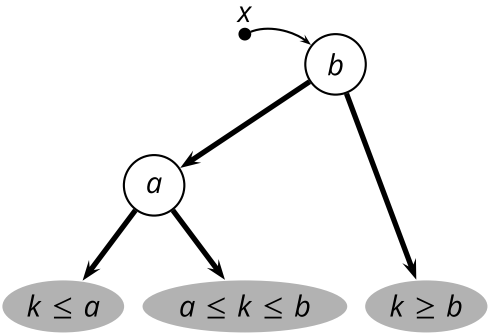
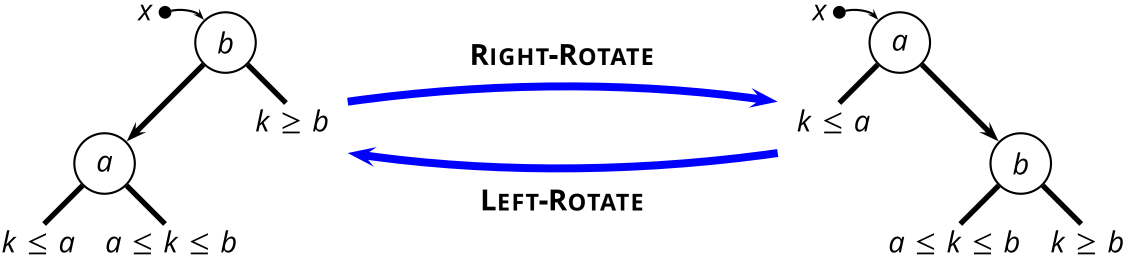



- $x$ = **Right-Rotate**$(x)$
- $x$ = **Left-Rotate**$(x)$

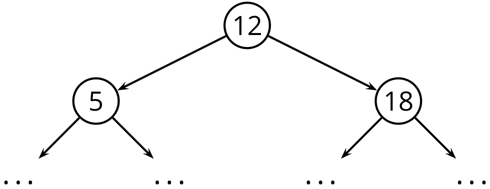
# Rotation




```
RIGHT-ROTATE(x)
  l = x.left
  x.left = l.right
  l.right = x
  return l
```

```
LEFT-ROTATE(x)
  r = x.right
  x.right = r.left
  r.left = x
  return r
```

# Root Insertion



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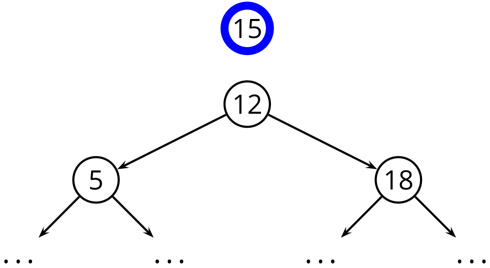

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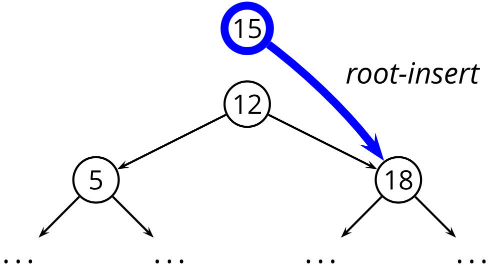


1. Recursively insert *z* at the root of the appropriate subtree (right)

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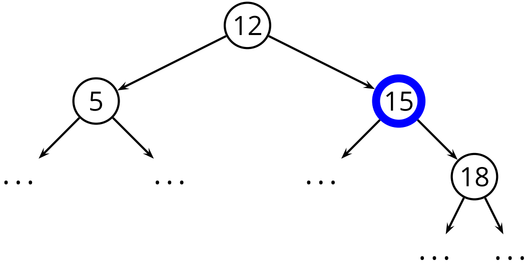


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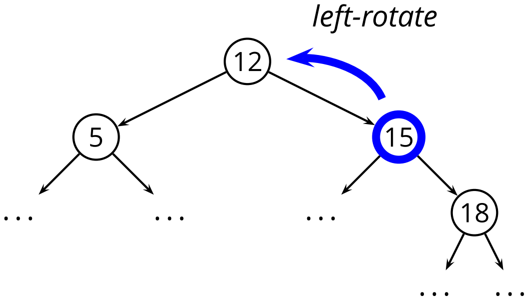


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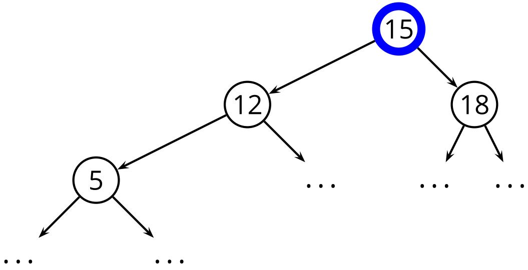


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# Root Insertion (2)

```
TREE-ROOT-INSERT(x, z)
if x = NIL
        return z
if z.key < x.key
        x.left = TREE-ROOT-INSERT(x.left, z)
        return RIGHT-ROTATE(x)
else x.right = TREE-ROOT-INSERT(x.right, z)
        return LEFT-ROTATE(x)
```

# Observation

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  - *amortized maintenance:* e.g., balancing a BST or resizing a hash table
    - relatively expensive but “amortized” operations
  - *optimized data structures:* a self-balanced data structure
    - guaranteed $O(\log n)$ complexity bounds

# Deletion

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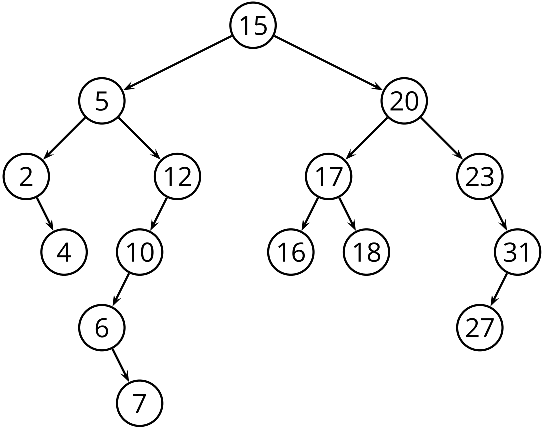

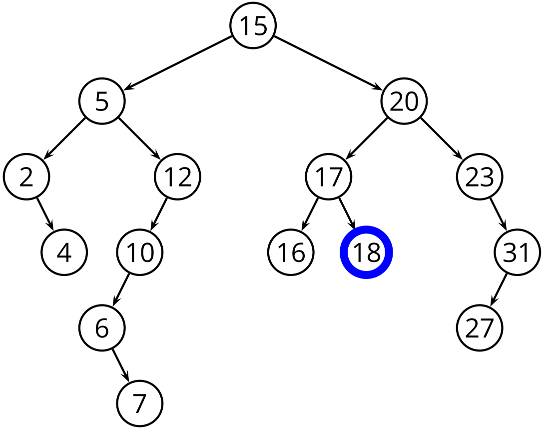


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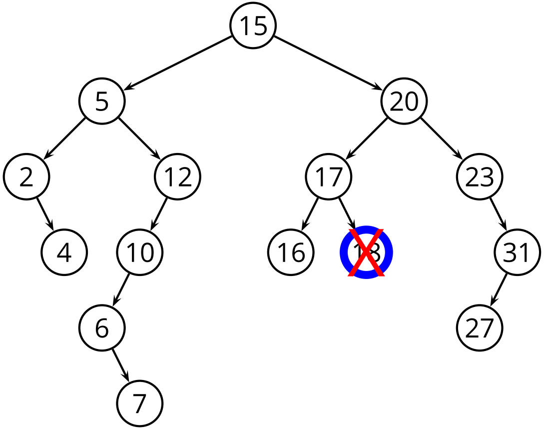


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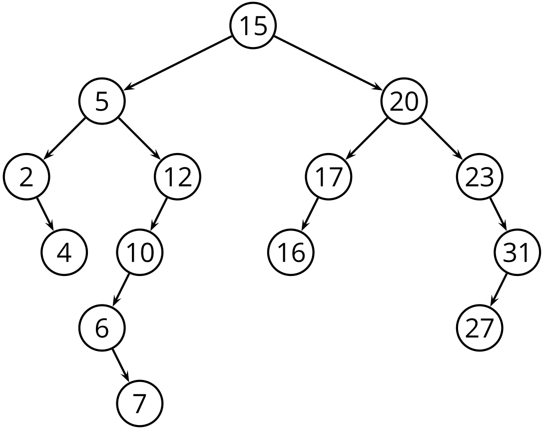


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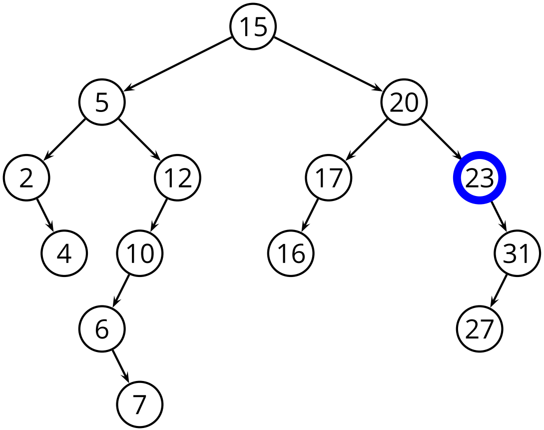


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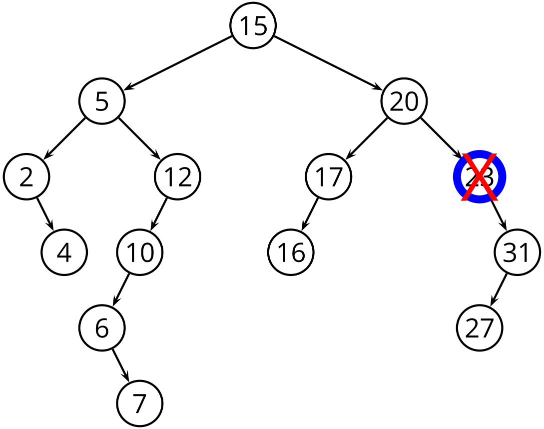


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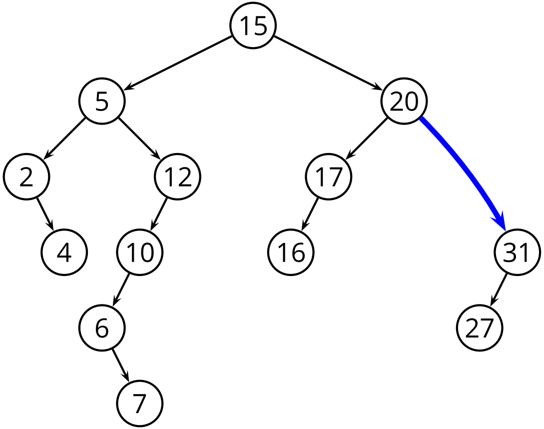


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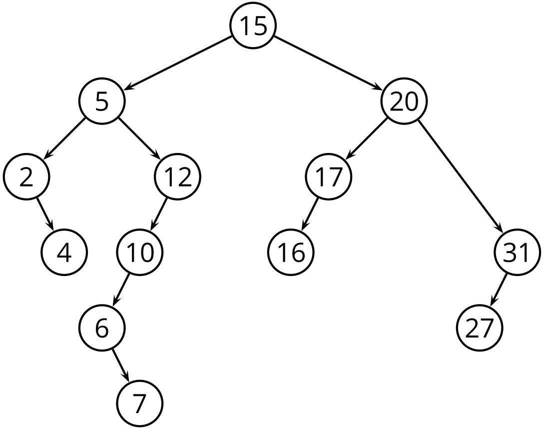


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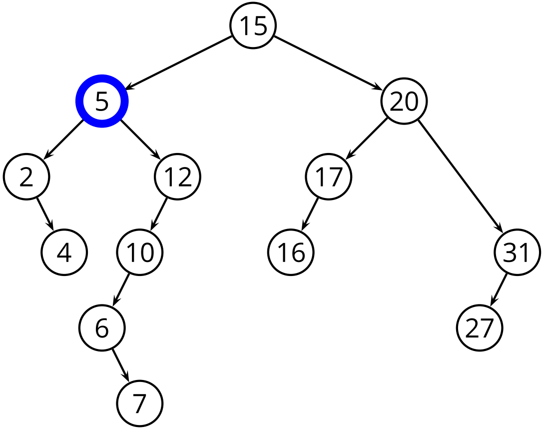


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3. $z$ has two children

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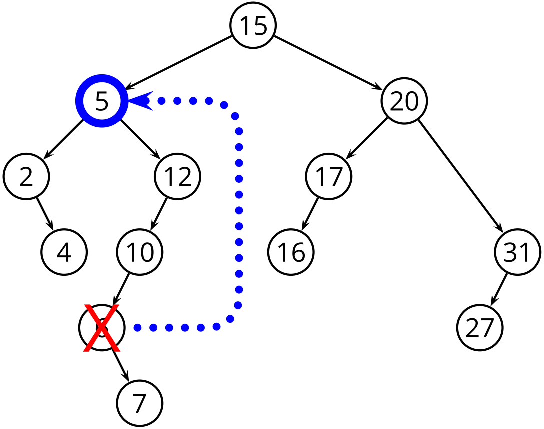


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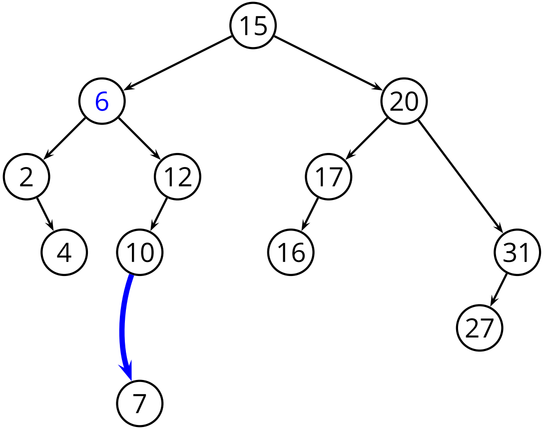


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   - replace $z$ with $y = \textbf{Tree-Successor}(z)$
   - remove $y$ (1 child!)
   - connect $y.parent$ to $y.right$

# Deletion (2)

```
TREE-DELETE(T, z)
if z.left = NIL or z.right = NIL
        y = z
else y = TREE-SUCCESSOR(z)
if y.left ≠ NIL
        x = y.left
else x = y.right
if x ≠ NIL
        x.parent = y.parent
if y.parent = NIL
        T.root = x
else if y = y.parent.left
                y.parent.left = x
        else y.parentright = x
if y ≠ z
        z.key = y.key
        copy any other data from y into z
```