Red-Black Trees (2)

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Recap on Red-Black Trees

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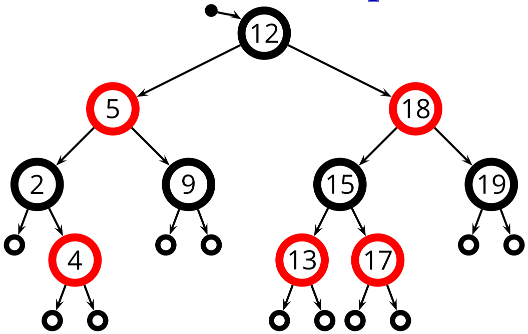

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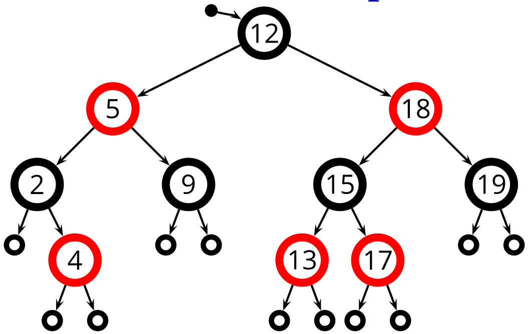

- *Red-black-tree property*

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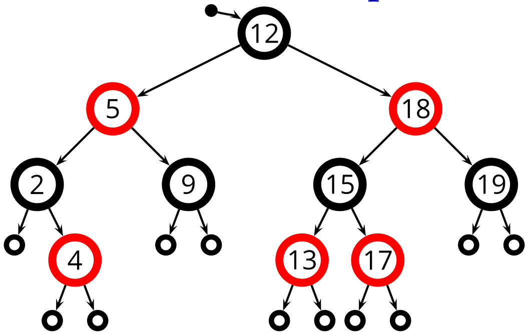


- *Red-black-tree property*

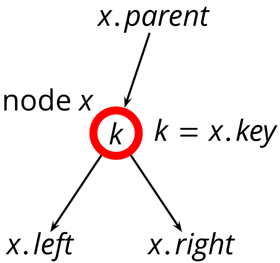
  1. every node is either **red** or **black**
  2. the root is **black**
  3. every (NIL) leaf is **black**
  4. if a node is **red**, then both its children are **black**
  5. for every node x, each path from x to its descendant leaves has the same number of **black** nodes $bh(x)$ (the *black-height* of x)

Recap on Red-Black Trees (2)

- *Implementation*
  - T represents the tree, which consists of a set of *nodes*
  - $T.root$ is the root node of tree T
  - $T.nil$ is the "sentinel" node of tree T

Nodes

- $x.parent$ is the parent of node x
- $x.key$ is the key stored in node x
- $x.left$ is the left child of node x
- $x.right$ is the right child of node x
- $x.color \in \{\text{RED}, \text{BLACK}\}$ is the color of node x



Recap on Deletion in Binary Trees

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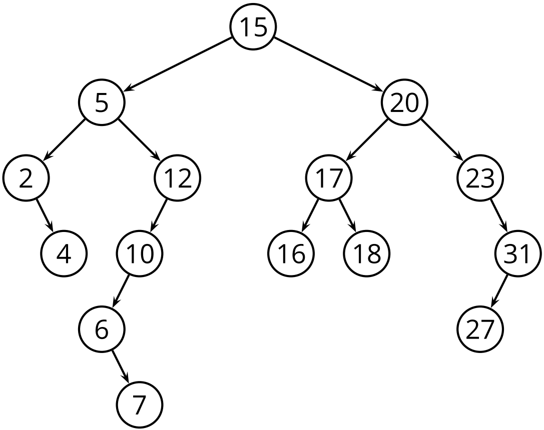

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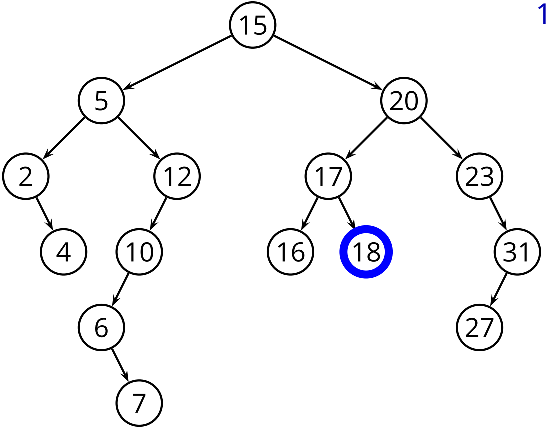

1. z has no children

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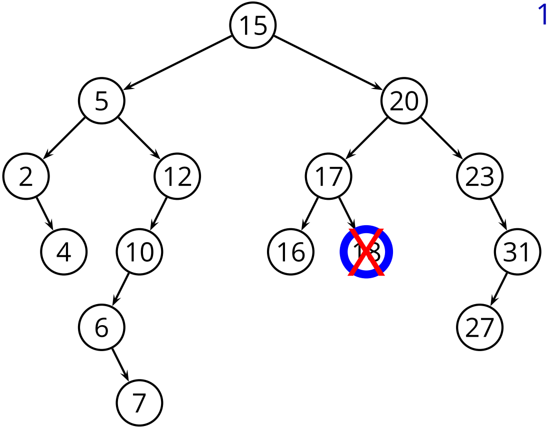

1. z has no children
   - simply remove z

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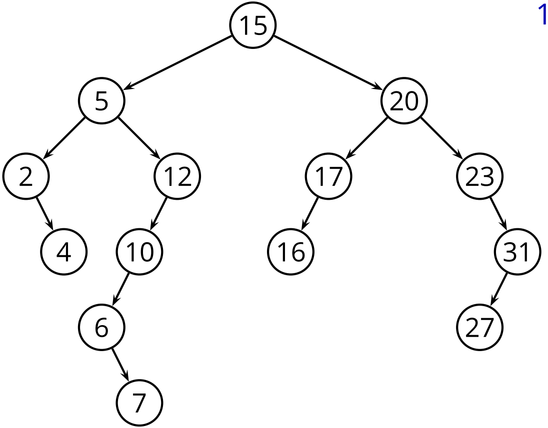

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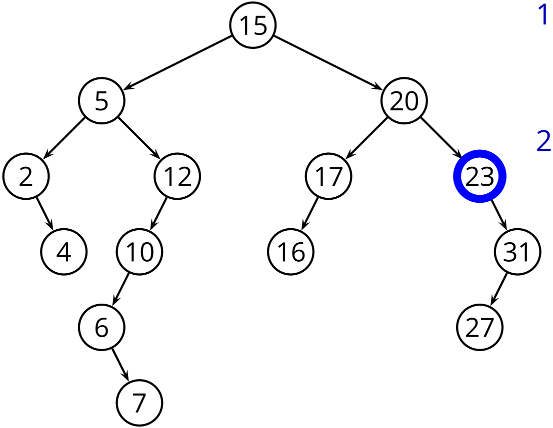

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2. *z* has one child

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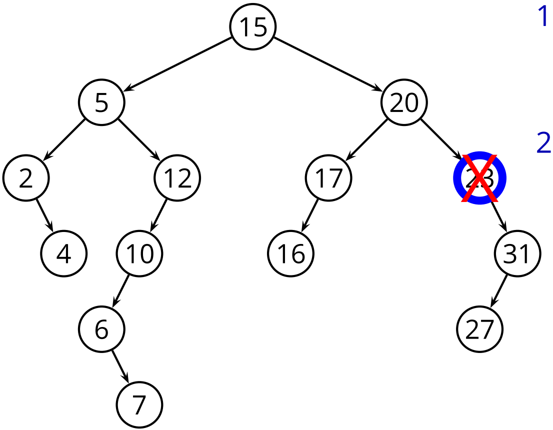

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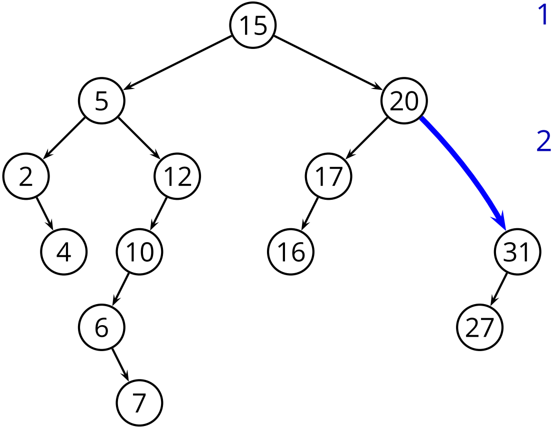

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   - connect $z.parent$ to $z.right$

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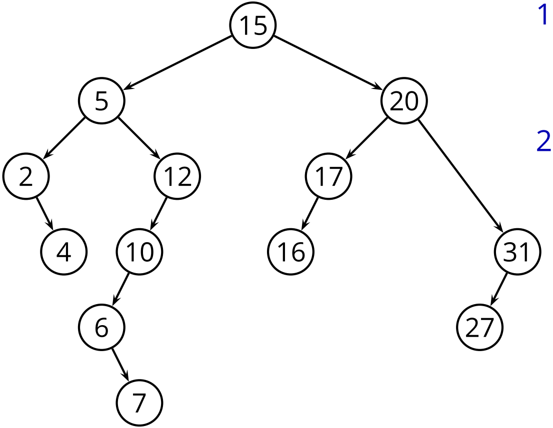

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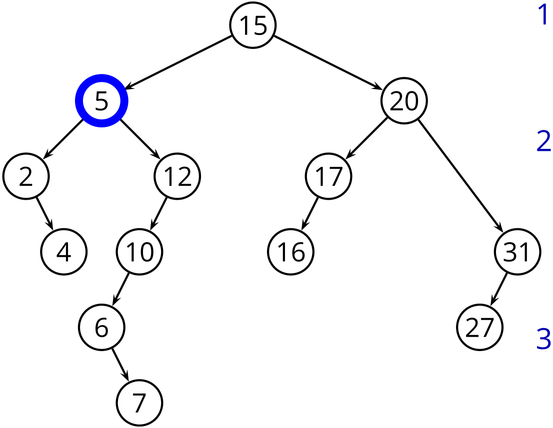

1. *z* has no children
   - simply remove *z*
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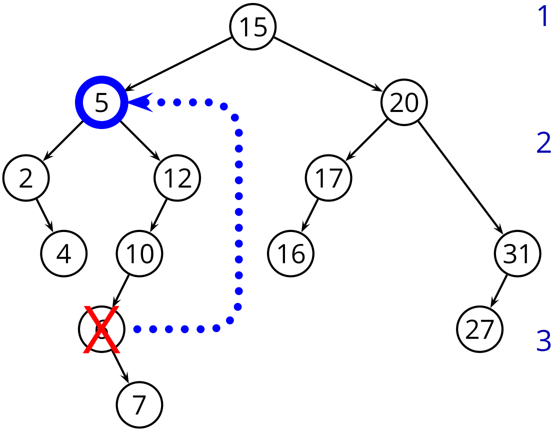
   - remove *z*
   - connect *z.parent* to *z.right*
3. *z* has two children

Recap on Deletion in Binary Trees



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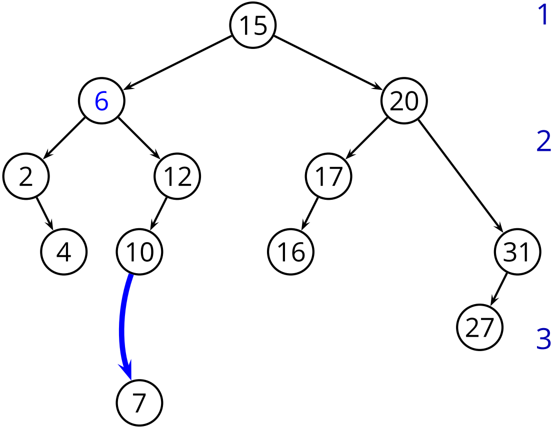
   - replace *z* with y = **Tree-Successor**(*z*)
   - remove *y* (1 child!)

Recap on Deletion in Binary Trees



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2. z has one child
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   - connect $z.parent$ to $z.right$
3. z has two children
   - replace z with y = **TREE-SUCCESSOR**(z)
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Red-Black Deletion

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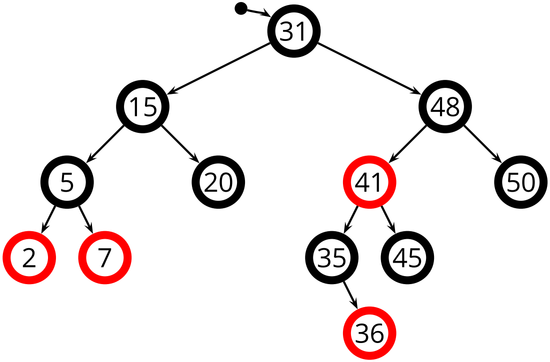

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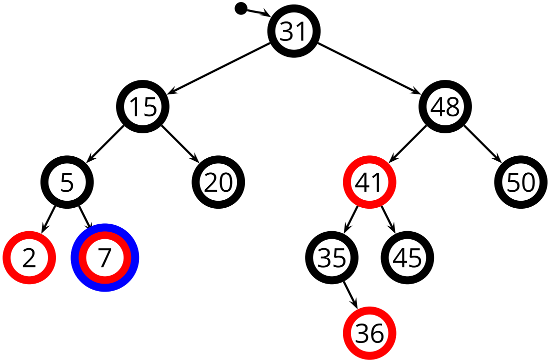

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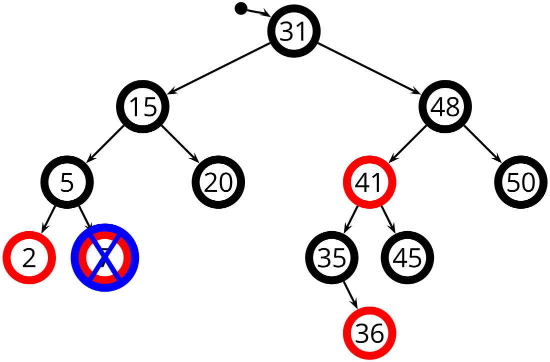

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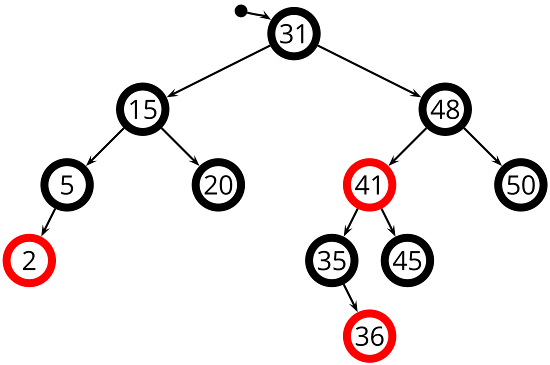

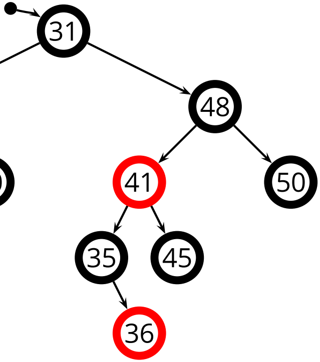

- A deleting a **red** *leaf* does not require any adjustment

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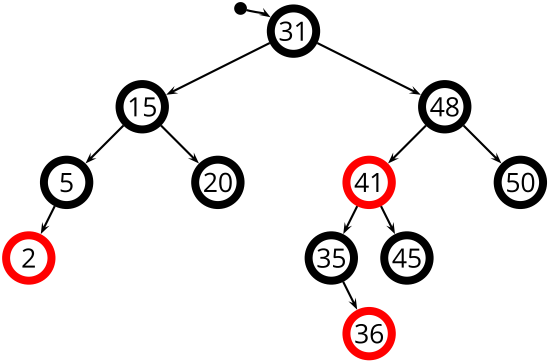

- A deleting a **red** *leaf* does not require any adjustment
  - the deletion does not affect the black height (property 5)

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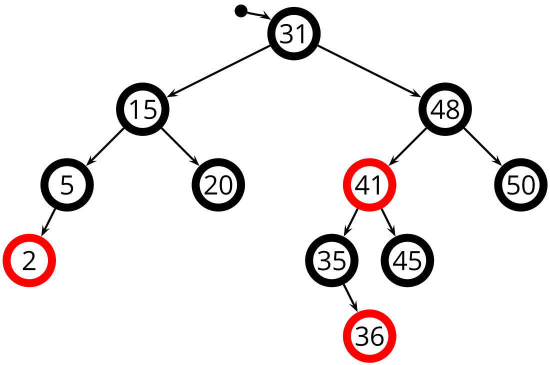
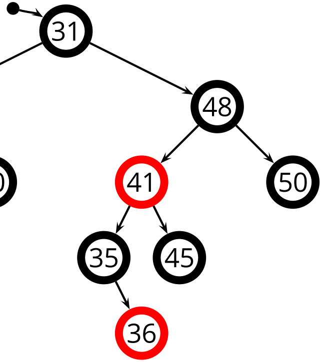


- A deleting a **red** *leaf* does not require any adjustment
  - the deletion does not affect the black height (property 5)
  - no two red nodes become adjacent (property 4)

Red-Black Deletion (2)



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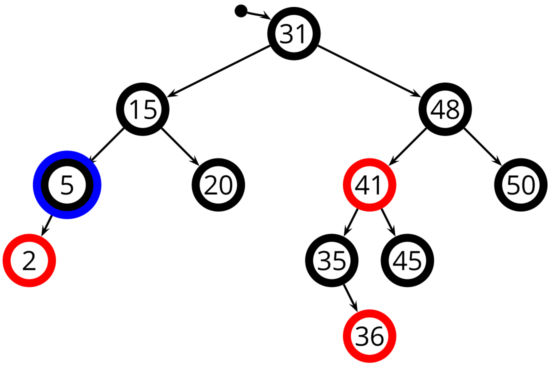

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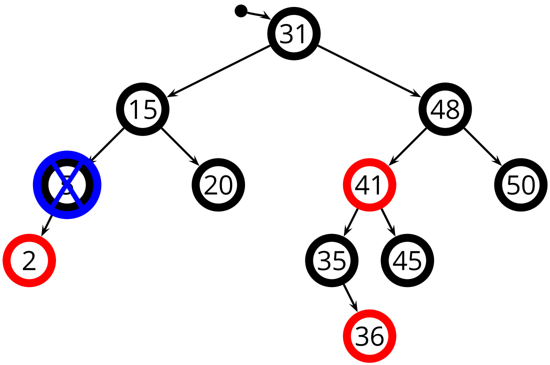

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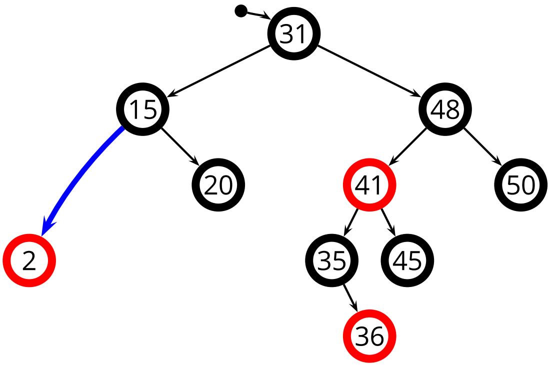

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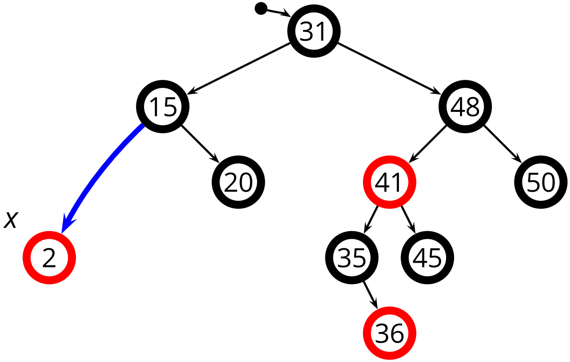

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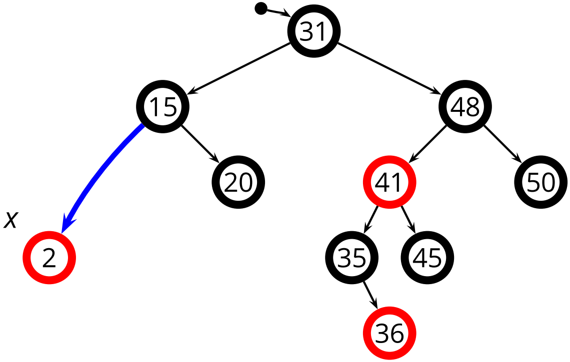

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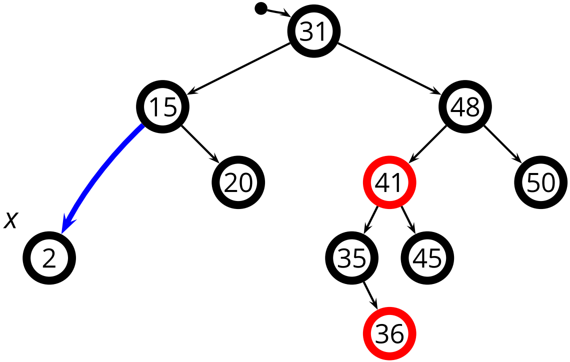

- Deleting a ***black*** *node* changes the balance of black-height in a subtree x
  - **RB-DELETE-FIXUP**(T, x) fixes the tree after a deletion
  - in this simple case: $x.color$ = BLACK

Fixup Conditions

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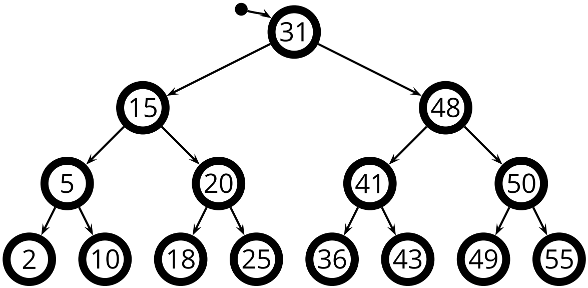
- ***Problem 3:*** we are removing y, which is black
  - violates red-black property 5 (same *black height* for all paths)

Red-Black Deletion (3)



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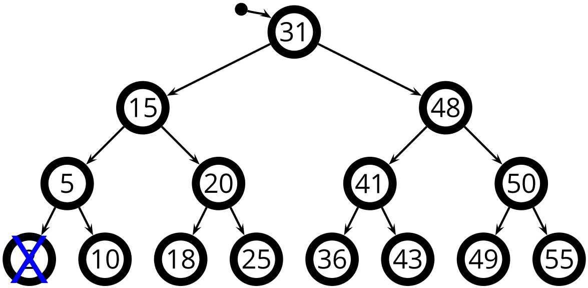

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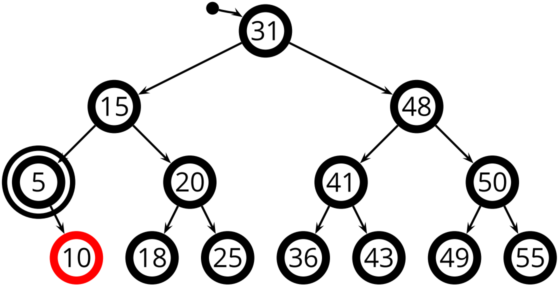

- *x* carries an *additional* ***black*** *weight*
  - the fixup algorithm pushes it up towards to root

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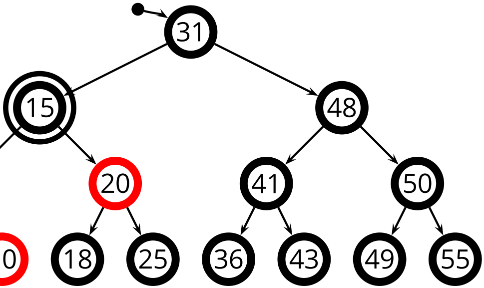

- *x* carries an *additional **black** weight*
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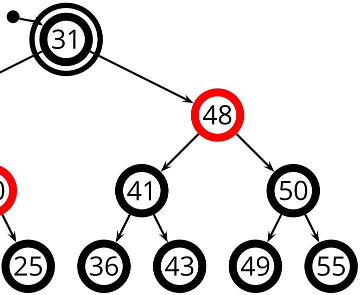

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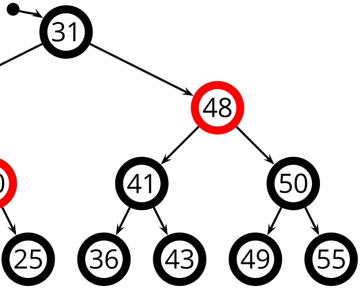

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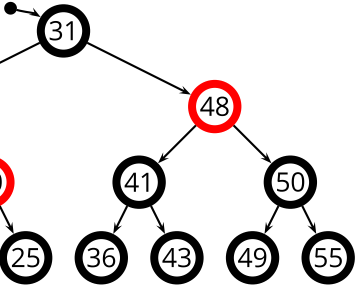
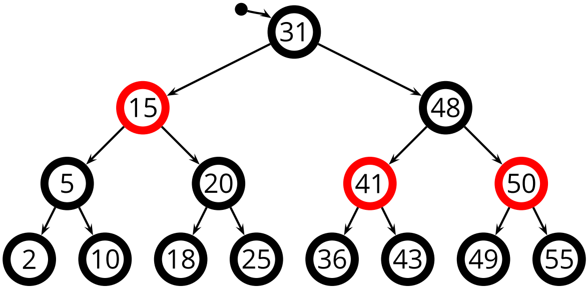
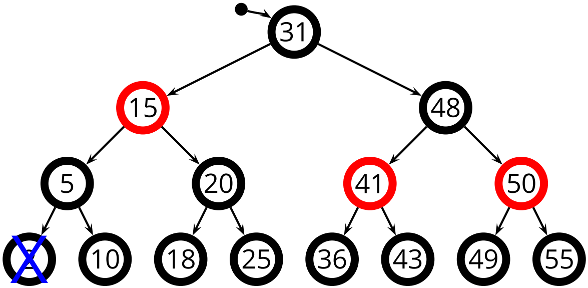


- *x* carries an *additional **black** weight*
  - the fixup algorithm pushes it up towards to root
- The *additional **black** weight* can be discarded if it reaches the *root*, otherwise. . .

Red-Black Deletion (4)



Red-Black Deletion (4)



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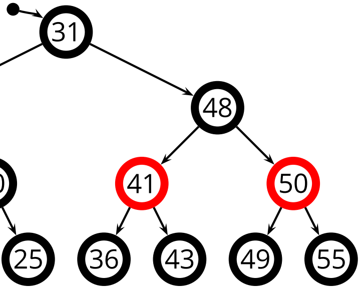

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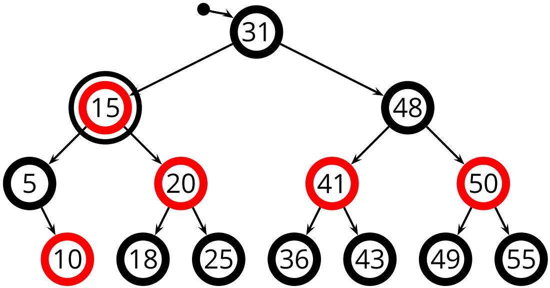

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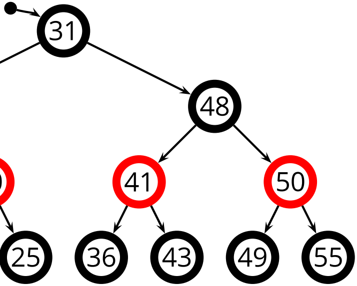
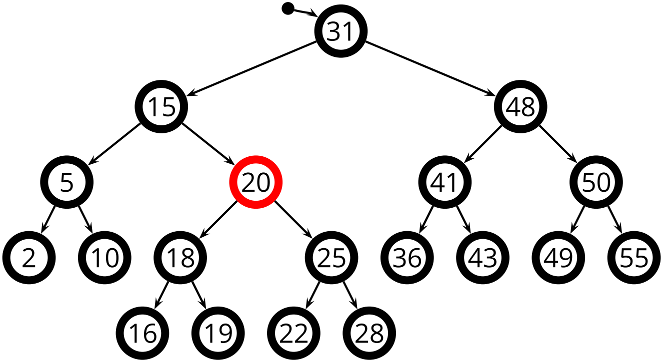


- The *additional **black** weight* can also stop as soon as it reaches a **red** node, which will absorb the extra **black** color

Red-Black Deletion (5)



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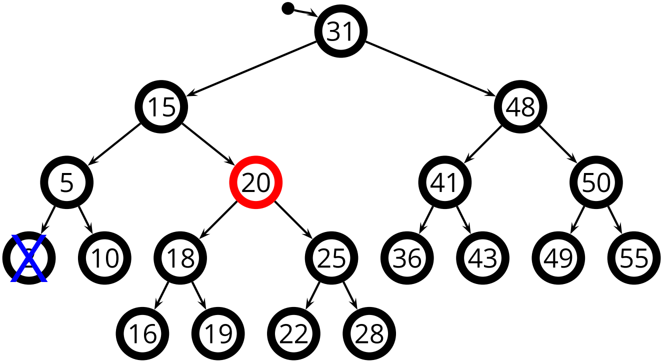

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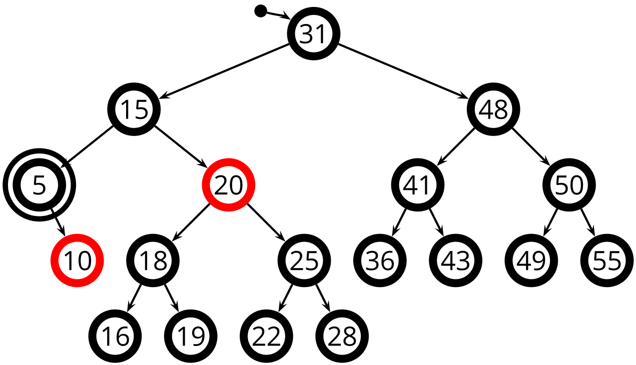

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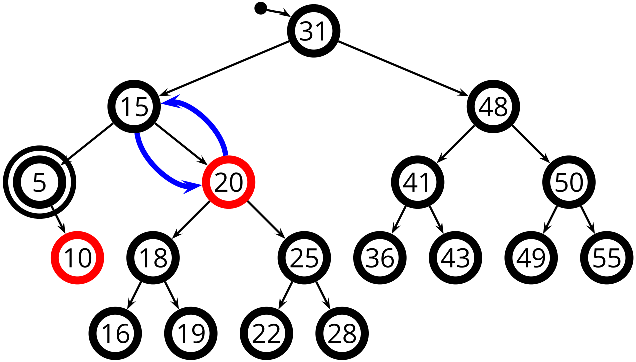

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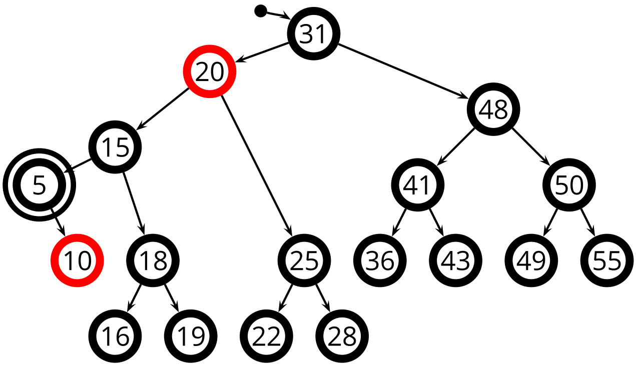

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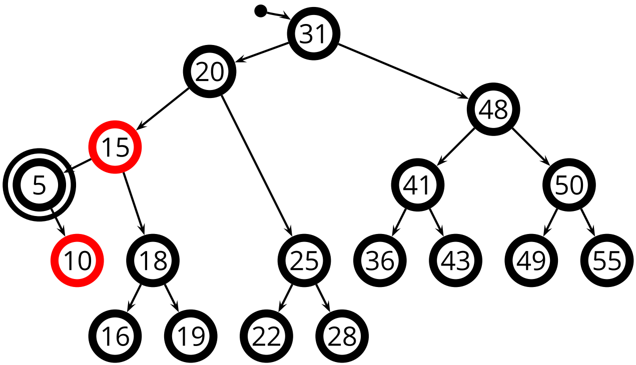

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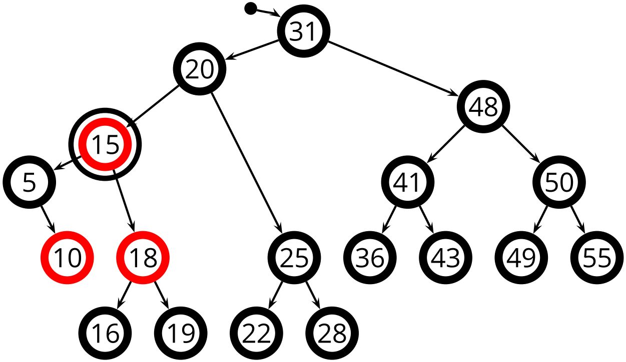

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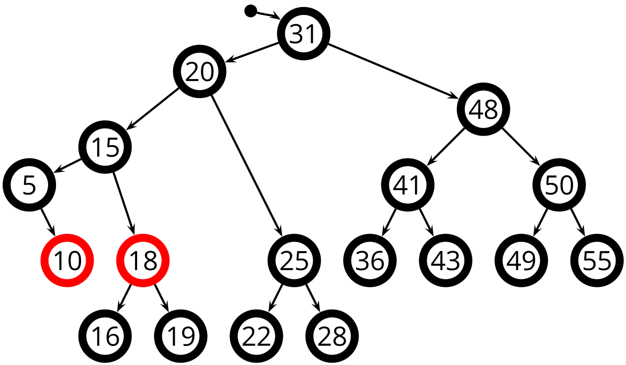

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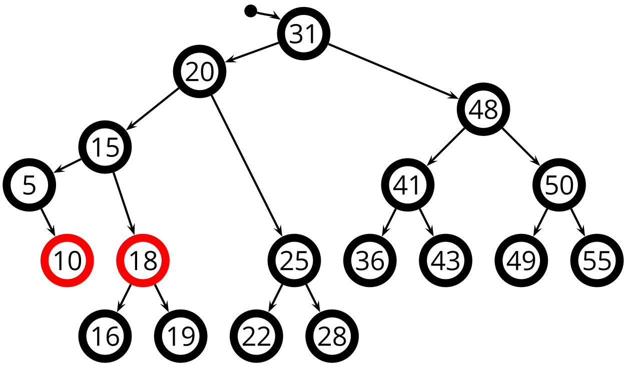

- In other cases where we can not push the additional black color up, we can apply appropriate rotations and color transfers that preserve all other red-black properties

Basic Fixup Iteration (1)

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Case 1

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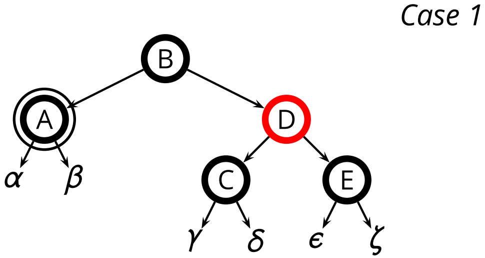

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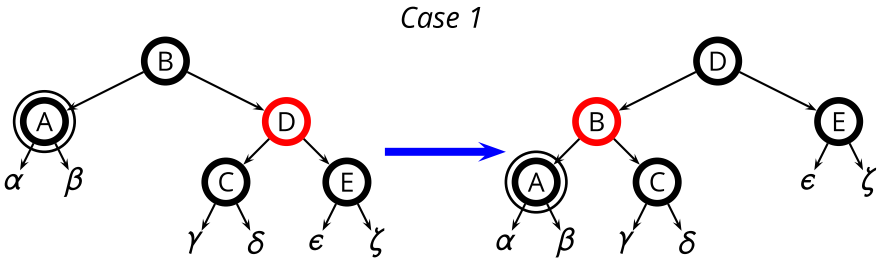

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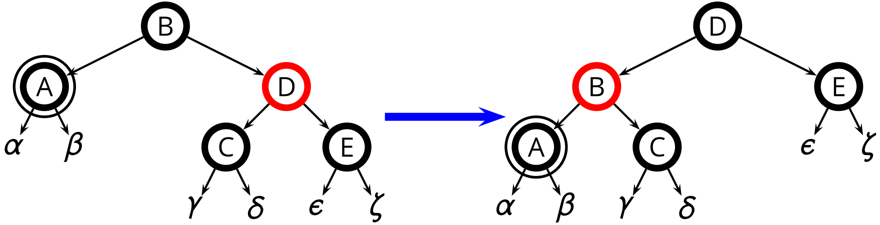
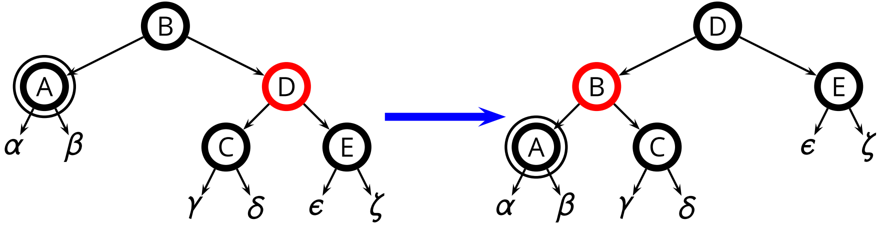


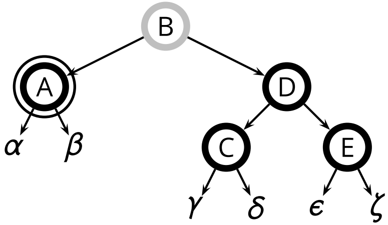
Case 2

Basic Fixup Iteration (1)

Case 1

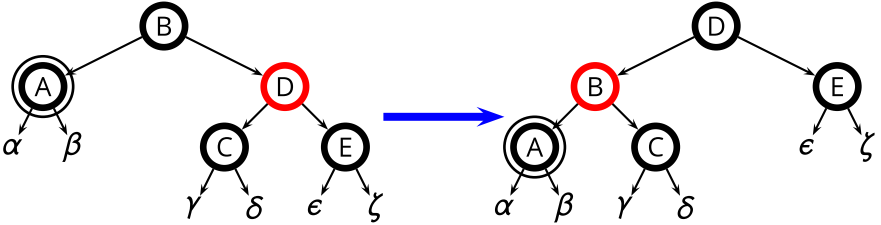


Case 2

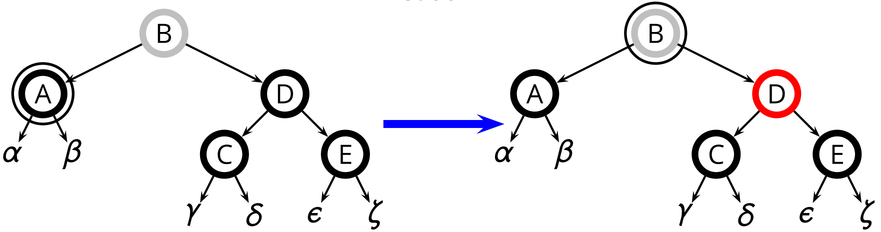


Basic Fixup Iteration (1)

Case 1



Case 2

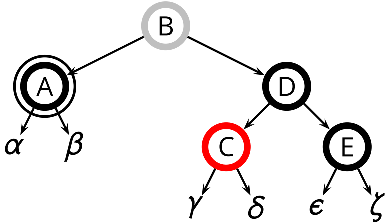


Basic Fixup Iteration (2)

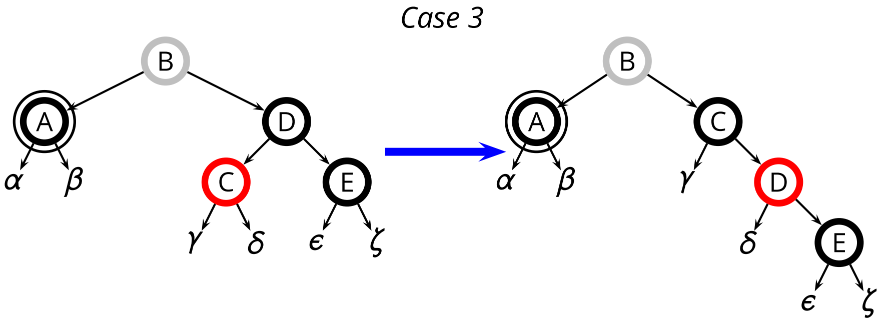
Case 3

Basic Fixup Iteration (2)

Case 3

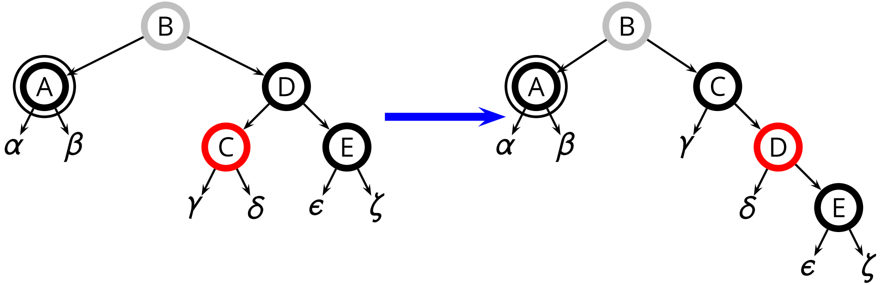


Basic Fixup Iteration (2)

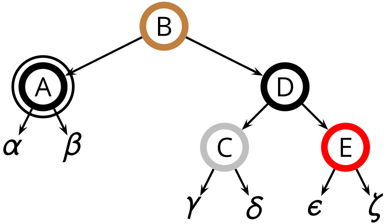


Basic Fixup Iteration (2)

Case 3

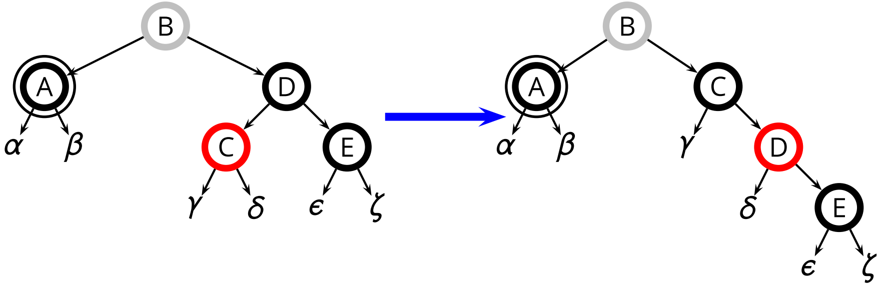


Case 4

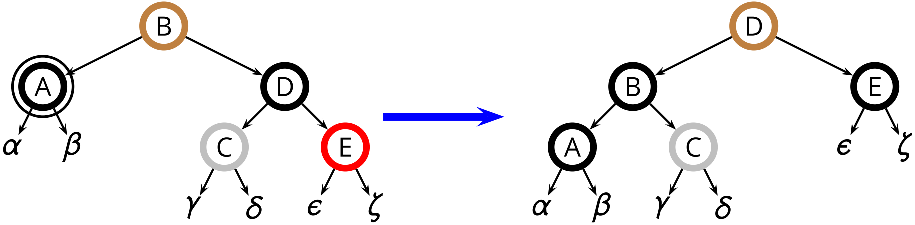


Basic Fixup Iteration (2)

Case 3



Case 4



Red-Black Delete Fixup

```
RB-DELETE-FIXUP(T, x)
while x ≠ T.root ∧ x.color = BLACK
    if x == x.parent.left
        w = x.parent.right
        if w.color == RED
            case 1...
        if w.left.color == BLACK ∧ w.right.color = BLACK
            w.color = RED                          // case 2
            x = x.parent
        else if w.right.color == BLACK
                case 3...
            case 4...
    else same as above, exchanging right and left
x.color = BLACK
```