Analysis of Insertion Sort

Antonio Carzaniga

Faculty of Informatics
Università della Svizzera italiana

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Outline

- Sorting
- Insertion Sort
- Analysis

- **Input:** a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$

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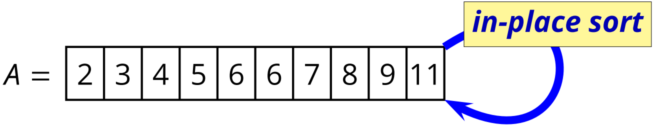

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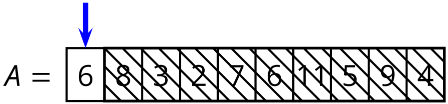

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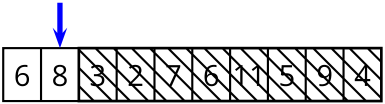

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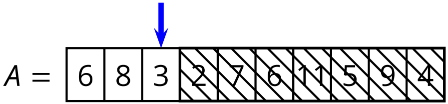

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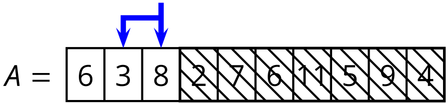

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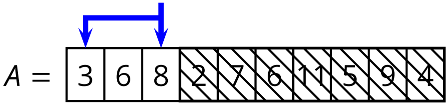

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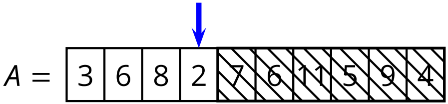

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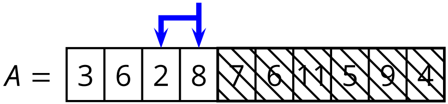

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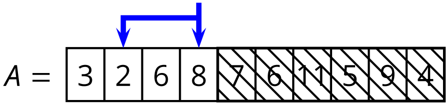

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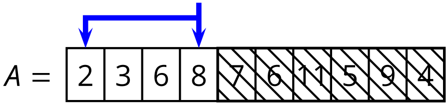

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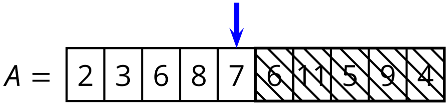

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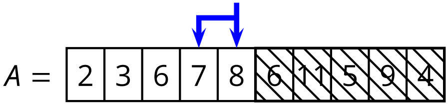

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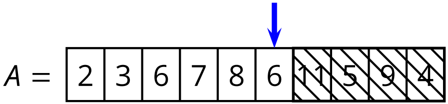

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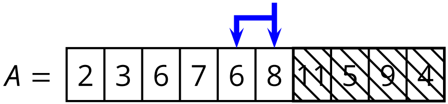

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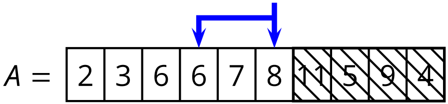

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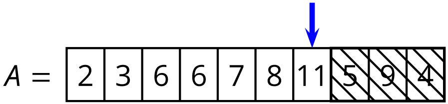

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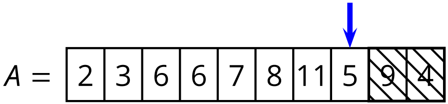

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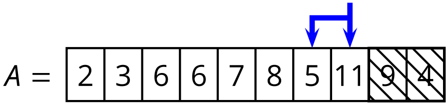

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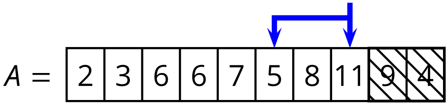

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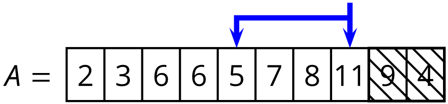

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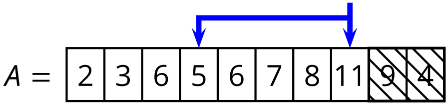

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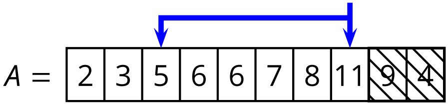

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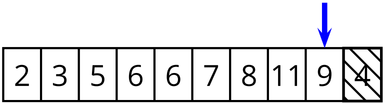

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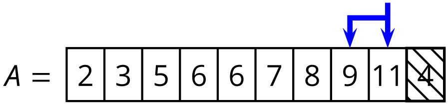

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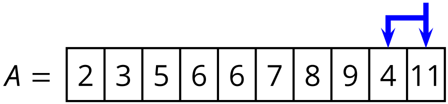

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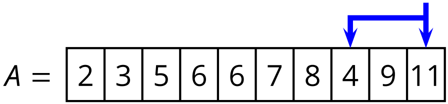

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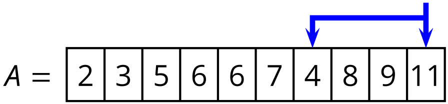

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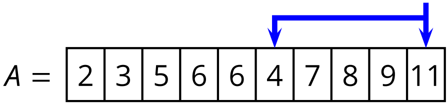

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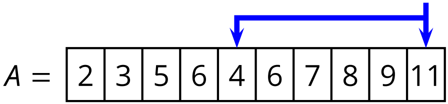

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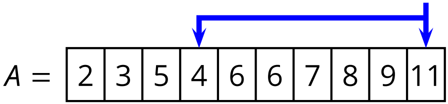

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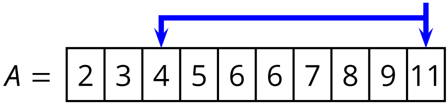

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Insertion Sort (2)

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
        j = j − 1
```

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- Is **INSERTION-SORT** *correct?*
- What is the time complexity of **INSERTION-SORT**?
- Can we do better?

Complexity of INSERTION-SORT

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Complexity of INSERTION-SORT

```
INSERTION-SORT(A)
1  for i = 2 to length(A)
2      j = i
3      while j > 1 and A[j − 1] > A[j]
4          swap A[j] and A[j − 1]
5          j = j − 1
```

- Outer loop (lines 1–5) runs exactly $n - 1$ times (with $n = \mathit{length}(A)$)
- What about the inner loop (lines 3–5)?
  - best, worst, and average case?

Complexity of INSERTION-SORT (2)

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- **Best case:**

Complexity of INSERTION-SORT (2)

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- **Best case:** the inner loop is *never* executed
  - what case is this?

- **Worst case:** the inner loop is executed exactly $j - 1$ times for every iteration of the outer loop
  - what case is this?

Complexity of INSERTION-SORT (3)

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- We want ***a formal proof of correctness***
  - does not seem straightforward. . .

The Logic of Algorithmic Steps

Example:

```
SORTTWO(A)
// A must be an array of 2 elements
if A[1] > A[2]
    t = A[1]
    A[1] = A[2]
    A[2] = t
```

Loop Invariants

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  - *C* must remain true *through* a loop
  - *C* is relevant to the problem definition: we use *C* at the end of a loop to prove the correctness of the result
- Then, we only need to prove that the algorithm terminates

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  - *the invariant must reflect the structure of the algorithm*
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  - it must be the basis to prove the correctness of the solution

- Proof of validity (i.e., that *C* is indeed a loop invariant): typical *proof by induction*
  - ***initialization:*** we must prove that

    the invariant C is true before entering the loop
  - ***maintenance:*** we must prove that

    if *C is true at the beginning of a cycle* ***then*** *it remains true after one cycle*

Loop Invariant for INSERTION-SORT

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- The main idea is to insert $A[i]$ in $A[1 \,..\, i-1]$ so as to maintain a *sorted subsequence* $A[1 \,..\, i]$

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- ***Invariant:*** (outer loop) *the subarray* $A[1 \,..\, i-1]$ *consists of the elements originally in* $A[1 \,..\, i-1]$ *in sorted order*

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- **Initialization:** $j = 2$, so $A[1\,..\,j-1]$ is the single element $A[1]$
  - $A[1]$ contains the original element in $A[1]$
  - $A[1]$ is trivially sorted

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- **Maintenance:** informally, if $A[1 \,..\, i-1]$ is a permutation of the original $A[1 \,..\, i-1]$ and $A[1 \,..\, i-1]$ is sorted (invariant), then *if* we enter the inner loop:
  - shifts the subarray $A[k \,..\, i-1]$ by one position to the right
  - inserts *key*, which was originally in $A[i]$ at its proper position $1 \leq k \leq i-1$, in sorted order

Loop Invariant for INSERTION-SORT (4)

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
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- **Termination: INSERTION-SORT** terminates with $i = length(A) + 1$; the invariant states that
  - $A[1 \mathinner{..} i-1]$ is a permutation of the original $A[1 \ldots i-1]$
  - $A[1 \mathinner{..} i-1]$ is sorted

  Given the termination condition, $A[1 \mathinner{..} i-1]$ is the whole A

  So **INSERTION-SORT** is *correct!*

Summary

- You are given a problem P and an algorithm A
  - P formally defines a *correctness* condition
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5. Prove that $X \wedge C \Rightarrow P$, which means that A is correct

Exercise: Analyze Selection-Sort

```
SELECTION-SORT(A)
n = length(A)
for i = 1 to n − 1
    smallest = i
    for j = i + 1 to n
        if A[j] < A[smallest]
            smallest = j
    swap A[i] and A[smallest]
```

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  - loop invariant?
- Complexity?
  - worst, best, and average case?

Exercise: Analyze Bubblesort

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BUBBLESORT(A)
for i = 1 to length(A)
    for j = length(A) downto i + 1
        if A[j] < A[j − 1]
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