Basic Elements of Complexity Theory

Antonio Carzaniga

Faculty of Informatics
Università della Svizzera italiana

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Outline

- Basic complexity classes
- Polynomial reductions
- NP-completeness

Polynomial Time

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Abstract Problems

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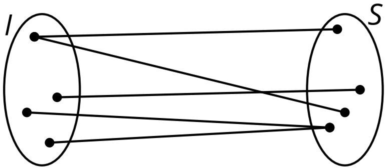
- An ***abstract problem*** *Q* is a binary relation between a set *I* of problem ***instances*** and a set *S* of ***solutions***



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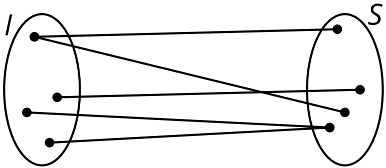

- A ***concrete problem*** Q is one where I and S are the set of binary strings $\{0, 1\}^*$
  - for all practical purposes, instances and solutions can be ***encoded*** as binary strings (i.e., mapped into $\{0, 1\}^*$)
  - we consider only sensible encodings...

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Primality Testing

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  $$G = (V = \{a, b, c, \ldots\}, E = \{(a, c), \ldots\}), a, z \longrightarrow a, c, \ldots, z$$

  - *input:* a graph G, a start vertex (a), and an end vertex (z)
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  Shortest path as a decision problem

  $$G = (V = \{a, b, c, \ldots\}, E = \{(a, c), \ldots\}), a, z, 10 \longrightarrow 1$$

  - *input:* a graph G, a start vertex (a), an end vertex (z), and a path length (10)
  - *output:* 1 if there is a path of (at most) the given length

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- An optimization problem is ***not much harder*** than the corresponding decision problem

  - having a solution to the decision problem does not give an immediate solution to the optimization problem
  - but we can typically use the decision problem as a subroutine in some kind of (binary) search to solve the corresponding optimization problem

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  - parsing a Java program
  - . . .

Verifying is Easy

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  - *Input:* A graph $G = (V, E)$ and a number K
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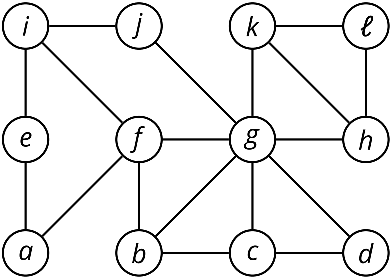

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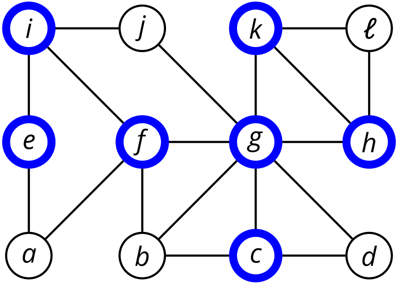

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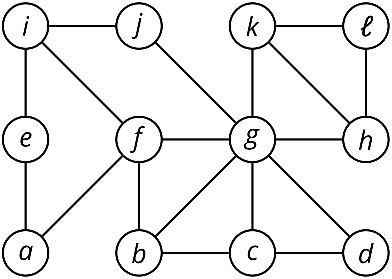

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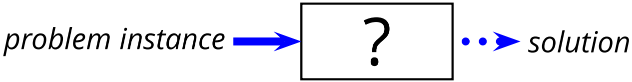

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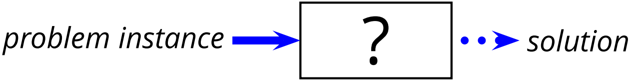

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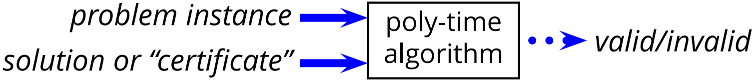

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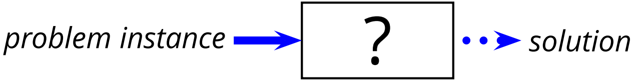

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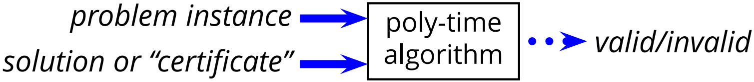

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- *polynomial-time solvable* $\Longrightarrow$ *polynomial-time verifiable*

$$P \subseteq NP$$

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- ***Finding a solution to a problem is believed to be inherently more difficult than verifying a given solution or a proof of a solution***

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Reduction

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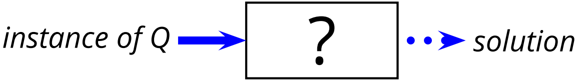

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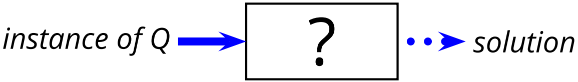

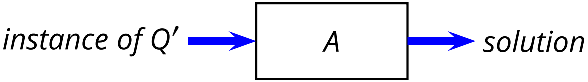

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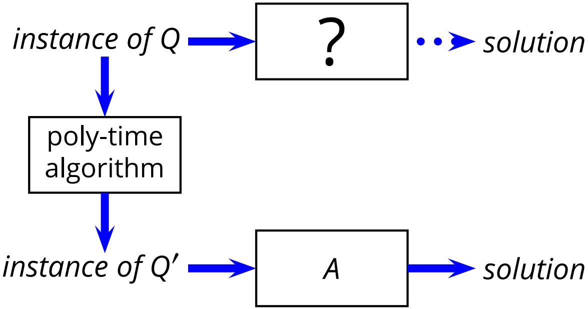

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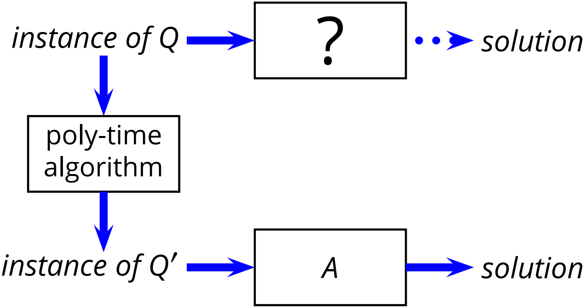

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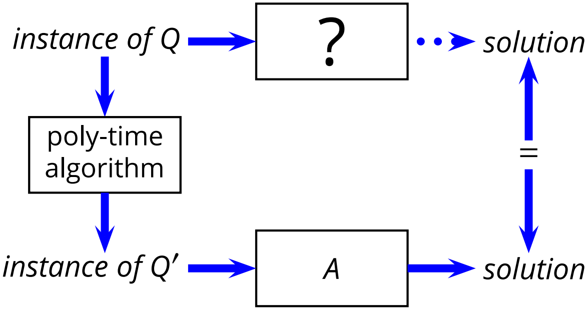

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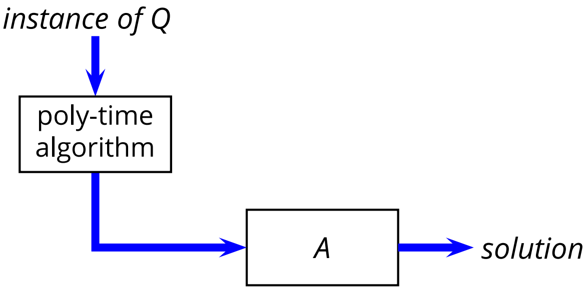
Reduction (2)

- Solution by polynomial-time reductions to a solvable problem



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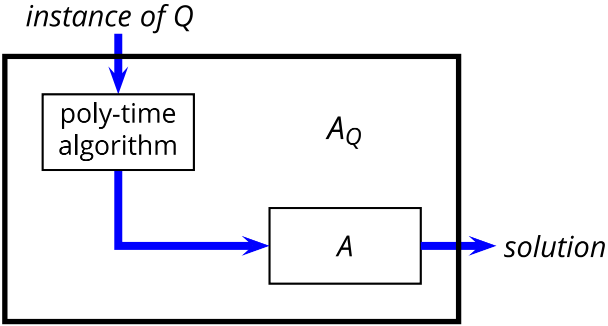

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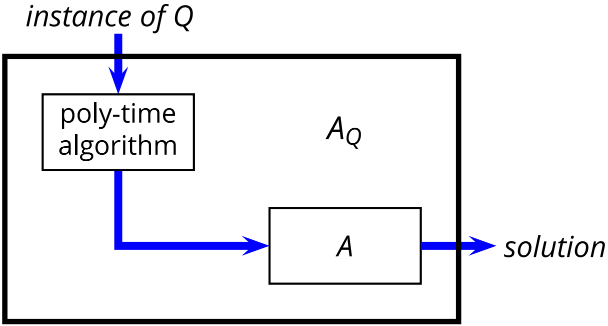

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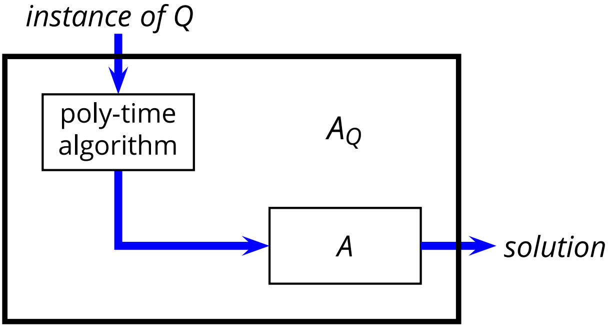

  - if A is polynomial-time, then of A_Q is also polynomial time
  - therefore if $Q' \in P$, then $Q \in P$

Example: 2-CNF-SAT

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- ***2-CNF-SAT problem***

  Input:

  - f is a Boolean formula of n (Boolean) variables $x_1, x_2, \ldots, x_n$
  - f is in *conjunctive normal form (CNF)*, so $f = C_1 \wedge C_2 \wedge \cdots \wedge C_k$
  - every *clause* C_i of f contains exactly *two* literals (a variable or its negation)

  Output: 1 iff f is satisfiable

  - there is an assignment of variables that satisfies f

  Example:

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2)$$

2-CNF-SAT to Implicative Form

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- **Example:**

$$(x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3)$$

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2-CNF-SAT to Graph Reachability

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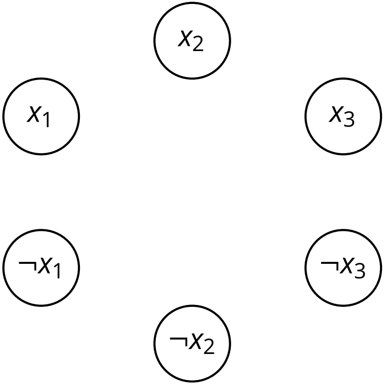

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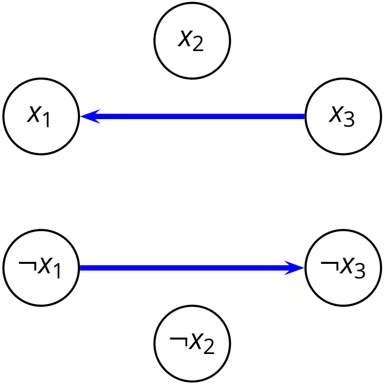

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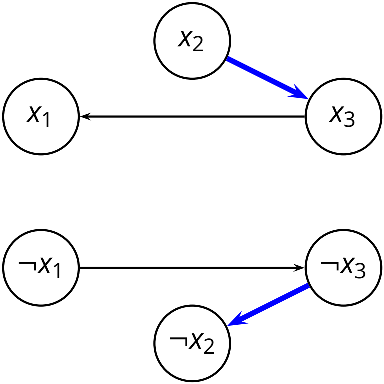

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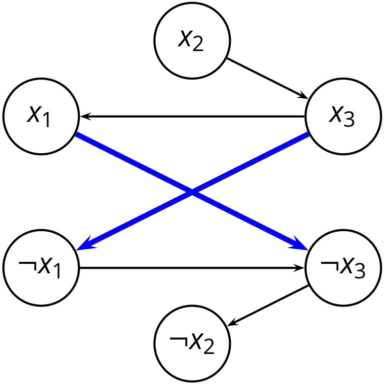

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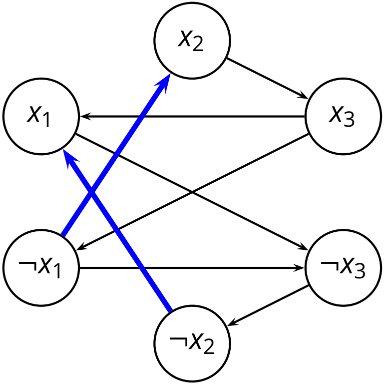

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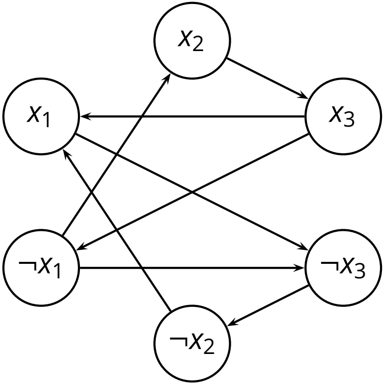

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if and only if
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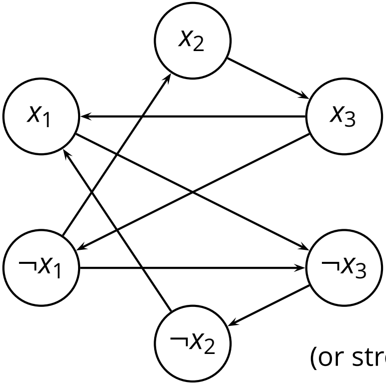

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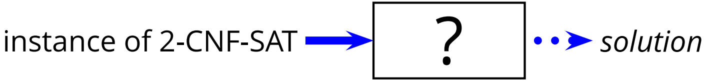
depth-first search

(or strongly connected components)

Reduction of 2-CNF-SAT

- 2-CNF-SAT $\in P$



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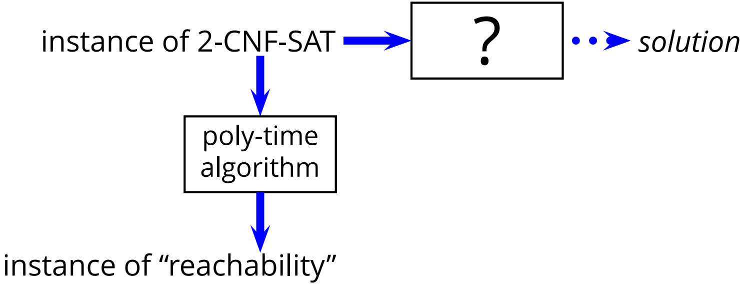

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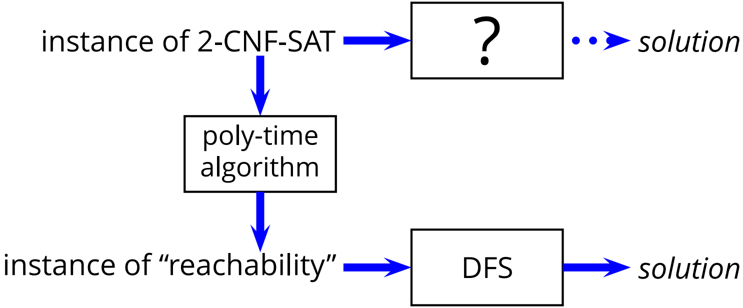
instance of 2-CNF-SAT
?
solution
poly-time algorithm
instance of "reachability"
DFS
solution

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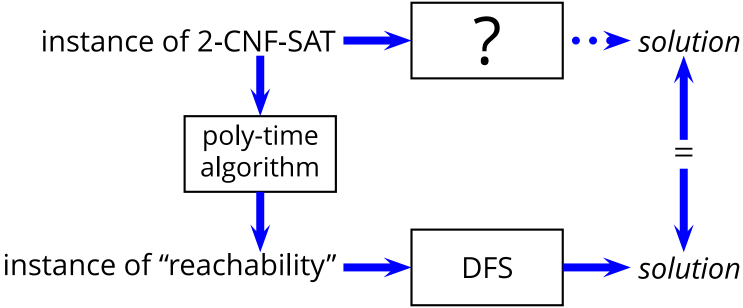

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- If Q' is NP-hard and *polynomial-time reducible* to Q'', then Q'' is NP-hard
- If Q' is NP-hard and *polynomial-time solvable*, then P $=$ NP
  - i.e., most researchers believe that there is no such Q'

The First NP-Complete Problem

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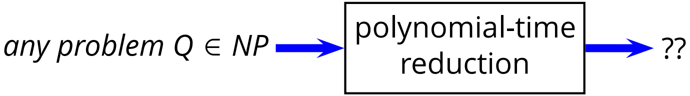
- Is there any NP-complete problem?



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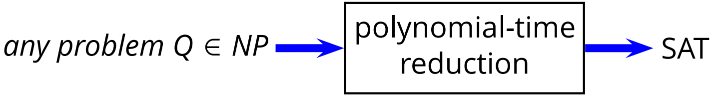

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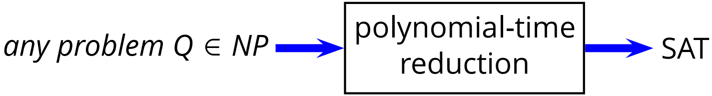

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- Many other problems were then proved NP-complete through polynomial reductions
  - e.g., SAT is polynomial-time reducible to the *longest path* problem
  - therefore, the *longest path* problem is also NP-complete